Global existence for the Einstein vacuum equations in wave coordinates

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Abstract

We prove global stability of Minkowski space for the Einstein vacuum equations in harmonic (wave) coordinate gauge for the set of restricted data coinciding with the Schwarzschild solution in the neighborhood of space-like infinity. The result contradicts previous beliefs that wave coordinates are "unstable in the large" and provides an alternative approach to the stability problem originally solved (for unrestricted data, in a different gauge and with a precise description of the asymptotic behavior at null infinity) by D. Christodoulou and S. Klainerman.

Using the wave coordinate gauge we recast the Einstein equations as a system of quasilinear wave equations and, in absence of the classical null condition, establish a small data global existence result. In our previous work we introduced the notion of a *weak null condition* and showed that the Einstein equations in harmonic coordinates satisfy this condition.The result of this paper relies on this observation and combines it with the vector field method based on the symmetries of the standard Minkowski space.

In a forthcoming paper we will address the question of stability of Minkowski space for the Einstein vacuum equations in wave coordinates for all "small" asymptotically flat data and the case of the Einstein equations coupled to a scalar field.

1 Introduction

The focus of this paper is the question of global existence and stability for the Einstein vacuum equations in "harmonic" (wave coordinate) gauge. The Einstein equations determine a 4-d manifold $\mathcal{M}$ with a Lorentzian metric g with vanishing Ricci curvature

$$R_{\mu\nu} = 0.$$

We consider the initial value problem: for a given a 3-d manifold Σ, with a Riemannian metric g_0, and a symmetric two-tensor k_0, we want to find a 4-d manifold $\mathcal{M}$, with a Lorentzian metric g satisfying the Einstein equations, and an imbedding $\Sigma \subset M$ such that g_0 is the restriction of g to Σ and k_0 is

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the second fundamental form of Σ in $\mathcal{M}$. The initial value problem is overdetermined which imposes compatibility conditions on the initial data: the constraint equations

$$R_0 - k_0{}^i_j\, k_0{}^j_i + k_0{}^i_i\, k_0{}^j_j = 0, \qquad \nabla^j k_{0ij} - \nabla_i\, k_0{}^j_j = 0, \qquad \forall i = 1, ..., 3.$$

Here R_0 is the scalar curvature of g_0 and ∇ is covariant differentiation with respect to g_0. The Einstein equations are invariant under diffeomorphisms. To have a working formulation one needs to eliminate this freedom by fixing a gauge condition or a system of coordinates.

While the Einstein equations are independent of the choice of a coordinate system, the existence of a special or preferred system of coordinates has been a subject of debate [Fo]. Historically, the first special coordinates were the *harmonic* coordinates (also referred to as *wave* coordinates in current terminology). These obey the equation $\Box_g\, x^\mu = 0$, $\mu = 0, 1, 2, 3$, where $\Box_g = \nabla_\alpha \nabla^\alpha$ is the geometric wave operator. Relative to the wave coordinates a Lorentzian metric g satisfies the *wave* coordinate condition if:[1]

$$g^{\alpha\beta}\partial_\beta g_{\alpha\mu} = \frac{1}{2} g^{\alpha\beta}\partial_\mu g_{\alpha\beta}, \qquad \forall \mu = 0, .., 3. \tag{1.1}$$

In this system of coordinates, the vacuum Einstein equations take the form of a system of quasilinear wave equations

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\, g_{\mu\nu} = \mathcal{N}_{\mu\nu}(g, \partial g), \qquad \forall \mu, \nu = 0, .., 3 \tag{1.2}$$

with a nonlinearity $\mathcal{N}(u, v)$ depending quadratically on v. In this particular *gauge* Choquet-Bruhat [CB1] was able to establish the existence of a globally hyperbolic development[2] of the Einstein vacuum equations starting with an arbitrary set of initial data prescribed on a 3-d space-like hypersurface and satisfying the constraint equations. While the result of Choquet-Bruhat and a later result of Choquet-Bruhat and Geroch [CB-G], establishing the existence of a maximal Cauchy development, constructs solutions for any given initial data set, it does not provide any information about the geodesic completeness of the obtained solution. In the language of the evolution equations these results only show the existence of "local in time" solutions.

The global results have proved to be by far more resistant. The outstanding global problem, which for a long time remained open, and was finally ingeniously solved by Christodoulou and Klainerman [C-K], was that of the stability of Minkowski space. In simplified language, it is the problem of constructing a global solution to the Einstein vacuum equations from the initial data, which is close to the Minkowski metric $m_{\mu\nu}$, and asymptotically approaching the Minkowski space. The initial data (Σ, g_0, k_0) for the problem of stability of Minkowski space is asymptotically flat, i.e., the complement of a compact set in Σ is diffeomorphic to the complement of a ball in $\mathbb{R}^3$, and there exists a system of coordinates (x_1, x_2, x_3) with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ such that for all sufficiently large r the metric[3] $g_{0ij} = (1+2M/r)\delta_{ij} + o(r^{-1-\sigma})$, and the second fundamental form $k_0 = o(r^{-2-\sigma})$ for some $\sigma > 0$. Here M is the mass, which by the positive mass theorem is positive unless the data is flat, see Schoen and Yau [S-Y] and Witten [Wi]. In addition, the data is required to satisfy a global smallness assumption, which makes sure that it is sufficiently close to the data $(\mathbb{R}^3, \delta, 0)$ for the Minkowski space.

[1] We shall use below the standard convention of summing over repeated indices and the notation $\partial_\alpha = \partial/\partial x^\alpha$

[2] For the definitions of global hyperbolicity and maximal Cauchy development see [H-E], [Wa]

[3] The stability result of [C-K] was proved for *strongly* asymptotically flat data $g_{0ij} = (1 + 2M/r)\delta_{ij} + o(r^{-3/2})$, $k_0 = o(r^{-5/2})$.

To understand some of the difficulties of the problem we recall that a generic system of quasilinear equations

$$\Box\phi_I = \sum_{|\alpha|\le|\beta|\le 2} A^{JK}_{I,\alpha\beta}\partial^\alpha\phi_J\partial^\beta\phi_K + \text{cubic terms} \tag{1.3}$$

allows solutions with smooth arbitrarily small initial data which blow up in finite time[4]. The key to global existence for such equations was the *null* condition found by Klainerman, [K2]. The small data global existence result for the equations satisfying the null condition was established in [C1], [K2]. The null condition manifests itself in special algebraic cancellations in the coefficients $A^{JK}_{I,\alpha\beta}$ of the quadratic terms of the equation.[5] It can be shown however, that the Einstein vacuum equations in wave coordinates do not satisfy the null condition. Moreover, Choquet-Bruhat [CB3] showed that even without imposing a specific gauge the Einstein equations violate the null condition.

These considerations led to the suggestion that the wave coordinates are not suitable for proving stability of Minkowski space. In fact, considering a second iterate of the equation (1.2), Choquet-Bruhat [CB2] argued that the Einstein vacuum equations are not stable in wave coordinates near the Minkowski solution. All these resulted in the belief that the wave coordinates are unstable in the large in the sense that a possible finite time blow up of solutions of the equation (1.2) is due to a coordinate singularity.

The global stability of Minkowski space had been proved by Christodoulou and Klainerman [C-K] who avoided the use of a preferred system of coordinates and instead relied on the invariant formulation of the Einstein equations with the choice of maximal time foliation (or the double null foliation in the new proof of Klainerman and Nicolo [K-N1]) and utilizing Bianchi identities for the curvature. The special structure of the quadratic terms plays a crucial part in the generalized energy estimates which form the backbone of the proof but the null condition can not be pointed out precisely.

A semiglobal stability result was also obtained in the work of Friedrich [Fr]. He used the conformal method to reduce the global problem to a local one. The approach is invariant and the special structure is again exploited implicitly.

In this paper we revisit the problem of global stability of Minkowski space in wave coordinates. More precisely, we consider the data[6] $(\mathbb{R}^3, g_0, k_0)$ with the metric g_0 coinciding with the spatial part of the Schwarzschild metric $g_S = (1+M/r)^4 dx^2$ in the region $r > 1 >> M$, vanishing second fundamental form k_0 for $r > 1$, and satisfying a global smallness assumption on $\mathbb{R}^3$. We prove that for this initial data the wave coordinate gauge is stable in the large: the reduced Einstein equations (1.2) has a global solution g defining a future causally geodesically complete space-time, [H-E]. The metric g in wave coordinates x^α, $\alpha = 0, .., 3$ approaches the Minkowski metric m: $\sup_{x\in\mathbb{R}^3} |g(t,x) - m| \to 0$ as $t \to \infty$.

The intuition behind this result is based on the observation that the Einstein vacuum equations in wave coordinates (1.2) satisfy the *weak null condition*. This notion was introduced in [L-R] for general quasilinear systems (1.3) and requires that the corresponding effective asymptotic system

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I = r^{-1} \sum_{n\le m\le 2} A^{JK}_{I,nm}(\partial_t - \partial_r)^n\Phi_J\, (\partial_t - \partial_r)^m\Phi_K, \qquad \Phi_I \sim r\phi_I \tag{1.4}$$

[4] This is in particular true for a *semilinear* equation $\Box\phi = (\partial_t\phi)^2$, [J1].

[5] E.g. $\Box\phi = (\partial_t\phi)^2 - |\nabla_x\phi|^2$ satisfies the null-condition.

[6] The existence of such data is guaranteed by the results of Corvino and Chrusciel-Delay, [Co], [C-D].

has global solutions for all small initial data.[7] Here,

$$A^{JK}_{I,nm}(\omega) = \sum_{|\alpha|=n,|\beta|=m} A^{JK}_{I,\alpha\beta}\hat\omega^\alpha\hat\omega^\beta, \qquad \hat\omega = (-1,\omega),\ \omega\in\mathbb{S}^2.$$

The classical null condition states that $A^{JK}_{I,nm}(\omega)\equiv 0$ and thus implies the weak null condition. The asymptotic system (1.4) arises as an approximation of (1.3) when one neglects the derivatives tangential to the outgoing Minkowski light cones, known to have faster decay. The asymptotic equation was introduced in [H1] to predict the time of a blow-up for scalar wave equations known to blow up in finite time, and was used in [L2] to find some other scalar wave equations for which the known blow-up mechanism was not present. Asymptotic systems played an important role in the analysis of the blow-up mechanisms in [A1].

In [L-R] we have shown that the asymptotic system generated by the Einstein equations in wave coordinates (1.2) has global solutions for all data. In this paper we consider the full nonlinear system (1.2). We should note that although the asymptotic system provides useful heuristics about the behavior of solutions, in particular the L^∞ decay of the first derivatives of various components of the metric g, it is barely used in our proof of the small data global existence result for the full nonlinear equation (1.2). While it is tempting to put forward a conjecture that, parallel to the result for the classical null condition [C1], [K2], the weak null condition guarantees the global existence result for small initial data, we can only argue that all known examples seem to confirm it. A simple example of an equation satisfying the weak null condition, violating the standard null condition and yet possessing global solutions for all data is given by the system

$$\Box\phi = w\cdot\partial^2\phi + \partial\psi\cdot\partial\psi, \qquad \Box\psi = 0, \qquad \Box w = 0 \tag{1.5}$$

Another example is provided by the equation $\Box\phi = \phi\Delta\phi$. The proof of a small data global existence result for this equation is quite involved, [L2] (radial case), [A3]. As we show in this paper the Einstein equations (1.2) is yet another example. Interestingly enough, at the level of an effective asymptotic system the Einstein equations can be modelled by the system (1.5).

The asymptotic behavior of null components of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ of metric g- the so called "peeling estimates"- was discussed in the works of Bondi, Sachs and Penrose and becomes important in the framework of asymptotically simple space-times (roughly speaking, space-times which can be conformally compactified), see also the paper of Christodoulou [C2] for further discussion of such space-times. Global solutions obtained in the work [C-K] were accompanied by very precise analysis of its asymptotic behavior although not entirely consistent with peeling estimates. However, global solutions obtained by Klainerman-Nicolo [K-N1] in the exterior[8] stability of Minkowski space were shown to possess peeling estimates for special initial data, [K-N2].

Our work is less precise about the asymptotic behavior and is focused more on developing a technically relatively simple approach allowing us to prove stability of Minkowski space in a physically interesting wave coordinate gauge In particular, we rely only on the standard Killing and conformal Killing vector fields of Minkowski space and do not construct almost Killing and conformal Killing vector fields adapted to the geometry of null cones of the solution g.

Our proof is based on generalized energy estimates combined with decay estimates. The generalized energy estimates are used with Minkowski vector fields $\{\partial_\alpha, \Omega_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha, S = x^\alpha\partial_\alpha\}$. For the

[7] For the precise definition see section 6.

[8] Outside of the domain of dependence of a compact set

equations satisfying the standard null condition uniform in time bounds on the generalized energies, combined with global Sobolev (Klainerman-Sobolev) inequalities, are sufficient to infer small data global existence. In our case however the generalized energies slowly grow in time (at the rate of t^{ε}) and need to be complemented by independent, not following from the global Sobolev inequalities, decay estimates. We derive the latter by direct integration of the equation along the characteristics. It is at this point that the intuition from the effective asymptotic system is most useful. We show that all components of the metric with exception of one decay at the rate of t^{-1}. The remaining component however decays only as $t^{-1+\varepsilon}$. Somewhat surprisingly, the glue that holds together such weak decay estimates and the generalized energy estimates is the wave coordinate condition (1.1).

In this paper we only prove the result for a restricted set of data coinciding with the Schwarzschild data outside of the ball of radius one.[9] This allows us to somewhat sidestep the problem of a long range effect of a gravitational field. Due to the inward bending of the light rays, solution arising from initial data coinciding with the Schwarzschild data outside of the ball of radius one will be equal to the Schwarzschild solution in the exterior of the Minkowski cone $r = t + 1$.

In our subsequent work we hope to be able to prove the stability of Minkowski space in wave coordinates for general data. In addition we hope to show that our method can be also used to treat the problem of small data global existence for the Einstein equations coupled to a scalar field.

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2 The main results and the strategy of the proof

We now formulate the main results of our paper. Our first result is global existence for the Einstein vacuum equations in wave coordinates.

Theorem 2.1. *Consider the reduced Einstein vacuum equations* [10]

$$\widetilde{\Box}_g h_{\mu\nu} = g^{\alpha\beta}\partial^2_{\alpha\beta} h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h), \qquad \forall \mu, \nu = 0, ..., 3, \tag{2.1}$$

where $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$ and the nonlinear term F is as in Lemma 3.2. We assume that the initial data $(g, \partial_t g)|_{t=0} = (g_0, g_1)$ are smooth, the Lorentzian metric is of the form

$$g_0 = -a^2 dt^2 + g_{0ij} dx^i dx^j$$

and

1) obey the wave coordinate condition

$$g^{\alpha\alpha'}\partial_\alpha g_{\alpha'\mu} = \frac{1}{2} g^{\alpha\alpha'}\partial_\mu g_{\alpha\alpha'}, \qquad \forall \mu = 0, ..., 3, \tag{2.2}$$

[9] Since the initial metric is always of the form $g_{ij} = (1 + 4M/r)\delta_{ij} + o(r^{-1})$ with $M > 0$, data coinciding with the Schwarzschild outside of a compact set is the closest analogue of compactly supported or rapidly decaying data usually considered in small data global existence results for nonlinear wave equations.

[10] In what follows we shall introduce the reduced wave operator $\widetilde{\Box}_g = g^{\alpha\beta}\partial^2_{\alpha\beta}$ and note that in wave coordinates $\widetilde{\Box}_g = \Box_g$, where $\Box_g \phi = |g|^{-1/2}\partial_\alpha\big(g^{\alpha\beta}|g|^{1/2}\partial_\beta\phi\big)$ is the geometric wave operator

2) satisfy the constraint equations

$$R_0 - |k_0|^2 + (trk_0)^2 = 0, \qquad \nabla^j k_{0ij} - \nabla_i trk_0 = 0, \quad \forall i = 1, ..., 3,$$

where R_0 is the scalar curvature of the metric g_{0ij}, and the second fundamental form $(k_0)_{ij} = -1/2a^{-1}g_{1ij}$.

3) we assume that the metric $(g_0)_{ij}$ coincides with the spatial part of the Schwarzschild metric g_s (in wave coordinates):

$$(g_0)_{ij} = \frac{r+2M}{r-2M}dr^2 + (r+2M)^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad r > 1$$

and $g_1 = 0$ for $r > 1$. Moreover, we assume that the lapse function $a^2(r) = (r-2M)/(r+2M)$ for $r > 1$ and $a(r) = 1$ for $r \le 1/2$

4) the data $(h_0, h_1) = (g_0 - m, g_1)$ verify the smallness condition

$$\varepsilon = \sqrt{E_N(0)} + M < \varepsilon_0, \tag{2.3}$$

where $N \ge 10$ and

$$E_N(t) = \sup_{0\le\tau\le t} \sum_{|I|\le N} \|\partial Z^I h(\tau, \cdot)\|^2_{L^2} \tag{2.4}$$

Here Z^I is a product of $|I|$ vector fields of the form ∂_i, $x_i\partial_j - x_j\partial_i$, $t\partial_i + x_i\partial_t$ and $t\partial_t + x^i\partial_i$. Then there exists a unique global smooth solution g with the property that for some constant C_N,

$$\begin{aligned} &E_N(t) \le 16\varepsilon^2(1+t)^{2C_N\varepsilon}, \\ &\|g_{\mu\nu}(t) - m_{\mu\nu}\|_{L^\infty_x} \le C_N\varepsilon(1+t)^{-1+C_N\varepsilon}. \end{aligned} \tag{2.5}$$

Remark 2.2. The existence of data satisfying the assumptions of the theorem follows from the work of [Co], [C-D], as argued in section 4.

A corollary of the above result is the global stability of Minkowski space for a restricted set of initial data.

Theorem 2.3. *Let $(\mathbb{R}^3, g_0, k_0)$ be the initial data set for the Einstein vacuum equations $R_{\mu\nu} = 0$. Assume that relative to some system of coordinates (x_1, x_2, x_3) the metric g_0 coincides with the spatial part of the Schwarzschild metric g_S outside the ball of radius one,*

$$g_0 = (1 + \frac{M}{r})^4 dx^2, \qquad r > 1,$$

while the second fundamental form k_0 vanishes for $r > 1$. In addition, we assume that relative to that system of coordinates g_0, M and k_0 satisfy the smallness condition

$$\sum_{0\le|I|\le N} \|\partial_x^I(g_0 - \delta)\|_{L^2(B_1)} + \sum_{0\le|I|\le N-1} \|\partial_x^I k_0\|_{L^2(B_1)} + M < \epsilon.$$

Then there exists a future causally geodesically complete[11] solution g together with a global system of wave coordinates with the property that the curvature tensor of g relative to these coordinates decays to zero along any future directed causal geodesic.

[11]For the definition see [H-E] and section 16 of this paper.

We now outline the strategy of the proof.

Remark 2.4. Throughout the paper we shall use the notation $A \lesssim B$ for the inequality $A \le CB$ with some large *universal* constant C. In our estimates we will make no distinction between the tensors $h_{\alpha\beta} = g_{\alpha\beta} - m_{\alpha\beta}$ and $H_{\alpha\beta} = m_{\alpha\alpha'} m_{\beta\beta'} (g^{\alpha\beta} - m^{\alpha\beta})$, since $H = -h + O(h^2)$ and the terms quadratic in h are lower order.

The continuity argument For the proof we let δ be any fixed number $0 < \delta < 1/2$. Let g be a local smooth solution of the reduced Einstein equations (2.1). We start with the weak estimate

$$E_N(t) \le 64\varepsilon^2 (1+t)^{2\delta} \tag{2.6}$$

By assumptions of the Theorem the estimate (2.6) holds for $t = 0$. Let $[0, T]$ be the largest time interval on which (2.6) still holds. We shall show that if $\varepsilon > 0$ is sufficiently small then on the interval $[0, T]$ the inequality (2.6) implies the same inequality with the constant 64 replaced by 16. It will then follow that the solution and the energy estimate (2.6) can be extended to a larger time interval $[0, T']$ thus contradicting the maximality of T. This will imply that $T = \infty$ and the solution is global. We will in fact prove that for a sufficiently small ε the stronger estimate (2.5) holds true on the interval $[0, T]$.

The global Sobolev inequality of Proposition 9.2 and the weak energy estimate (2.6) imply the pointwise decay estimates:

$$\sum_{|I| \le N-2} |\partial Z^I h(t,x)| \le \frac{C\varepsilon (1+t)^{\delta}}{(1+t+r)(1+|t-r|)^{1/2}}, \qquad r = |x| \tag{2.7}$$

From the assumption that the constant $\delta < 1/2$ we derive the following weak decay estimates

$$|\partial Z^I h(t,x)| \le C\varepsilon (1+t+r)^{-1/2-\gamma} (1+|t-r|)^{-1/2-\gamma}, \qquad |I| \le N-2 \tag{2.8}$$

with some fixed constant $\gamma > 0$. The weak decay estimates (2.8) will lead to much stronger decay estimates in Theorem 14.1. In turn, using the stronger decay estimates in Theorem 14.1 we will be able to obtain stronger energy estimates in Theorem 15.1, i.e. (2.5). These in particular will enable us to show that the estimate (2.6) holds globally in time and conclude the proof. We remark that in the course of the proof all constants will be independent of $\varepsilon > 0$ but they will depend on a lower bound for $\gamma > 0$ (and hence on an upper bound for $\delta < 1/2$).

As described above, the proof is a direct consequence of three results. First is the global Sobolev inequality of Proposition 9.2, introduced by S. Klainerman [K1], giving decay estimates in terms of energy estimates for the generators of the Lorentz group. The second ingredient is the improved decay estimates in Theorem 14.1. The final component is the energy estimates in Theorem 15.1 which rely on the improved decay estimates.

Weak decay estimates. As pointed out above we may start by assuming the weak decay estimate (2.8). Furthermore, since the solution $g = m + h$ coincides with the Schwarzschild solution of mass $M \le \varepsilon$ in the region $r \ge t+1$, we have

$$|Z^I h(t,x)| \lesssim \varepsilon (1+r+t)^{-1}, \qquad \text{when} \quad |x| = t+1 \tag{2.9}$$

Hence integrating (2.8) from the light cone, where (2.9) holds, we get

$$|Z^I h(t,x)| \lesssim \varepsilon (1+r+t)^{-1/2-\gamma} (1+|t-r|)^{1/2-\gamma}, \tag{2.10}$$

Since the vector fields span the tangent space of the outgoing light cones $r-t=q$ we infer, with $\bar\partial$ denoting the derivatives tangential to the cones, that

$$(2.11)\qquad |\bar\partial Z^I h|\lesssim \varepsilon(1+r+t)^{-3/2-\gamma}(1+|t-r|)^{1/2-\gamma},$$

This means that, close to the light cone $t=r$, derivatives tangential to the forward light cones decay quite a bit better than the expected decay rate from (2.8) for a generic derivative.

Wave coordinate condition. As we shall see below certain components of the tensor h decay faster than others. This can be seen upon introduction of a null frame of vector fields $L=\partial_t+\partial_r$, $\underline{L}=\partial_r-\partial_t$ and S_1, S_2: two orthonormal vectors tangential to the sphere of radius r in $\mathbf{R}^3$. The first improved estimates come from the wave coordinate condition (2.2). Writing $g_{\alpha\beta}=m_{\alpha\beta}+h_{\alpha\beta}$ we obtain from (2.2) that

$$m^{\alpha\beta}\partial_\alpha h_{\beta\mu}=\partial_\mu m^{\alpha\beta}h_{\alpha\beta}+O(h\,\partial h)$$

In particular, contracting with a vector field $T\in\mathcal{T}=\{L,S_1,S_2\}$ and using that for any symmetric 2-tensor k, $m^{\alpha\beta}k_{\alpha\beta}=-k_{L\underline{L}}+\delta^{AB}k_{AB}$, implies that we can express the transversal derivative $\partial_{\underline{L}}$ of certain components of h in terms of the tangential derivatives that decay better and a quadratic term

$$|(\partial h)_{LT}|\le|\bar\partial h|+|h|\,|\partial h|\lesssim\varepsilon(1+t+r)^{-1-2\gamma},\qquad |h_{LT}|\lesssim\varepsilon(1+|t-r|)(1+t+r)^{-1}$$

Even though the estimate above does not give a better decay rate for all components of h it gives the decay exactly for those components which, as it turns out, control the geometry, i.e., they lead to stronger energy and decay estimates.

The above estimates will be sufficient to obtain improved estimates for the lowest order energy of h. However, in order to get estimates for the energy of $Z^I h$ we commute the vector fields Z through the equation for h. This generates additional commutator terms. The main commutator terms are controlled with the help of the following additional estimate from the wave coordinate condition:

$$(2.12)\quad |(\partial h)_{LT}|+|(\partial Zh)_{LL}|\le\varepsilon(1+t+r)^{-1-2\gamma},\qquad |h_{LT}|+|(Zh)_{LL}|\le\varepsilon(1+|t-r|)(1+t+r)^{-1}$$

We now describe derivation of the stronger decay and energy estimates.

Stronger decay estimates. We rely on the following decay estimate for the wave equation on a curved background [12]:

$$(2.13)\quad \|(1+t+r)\partial\phi(t,\cdot)\|_{L^\infty}\le C\int_0^t(1+\tau)\|\widetilde{\square}_g\phi(\tau,\cdot)\|_{L^\infty}\,d\tau + C\sup_{0\le\tau\le t}\sum_{|I|\le1}\|Z^I\phi(\tau,\cdot)\|_{L^\infty}+C\int_0^t\sum_{|I|\le2}(1+\tau)^{-1}\|Z^I\phi(\tau,\cdot)\|_{L^\infty}\,d\tau$$

The estimate (2.13) will be applied to the components of the tensor h. The term Z^Ih on the right hand side of the estimate will be controlled with the help of the weak decay estimates, and thus the decay rate of h will be determined in terms of decay of $\widetilde{\square}_g h$. The $L^\infty-L^\infty$ estimate (2.13) does not rely on the fundamental solution as does the more standard L^1-L^∞ type estimate. This estimate was used [L1] in the constant coefficient case and here we establish it in the variable coefficient case

[12] Recall that the reduced wave operator $\widetilde{\square}_g=g^{\alpha\beta}\partial^2_{\alpha\beta}$.

only under the assumption of the weak decay of all of the components of the metric g and the stronger decay of the components of g controlled by the wave coordinate condition. This analysis is by itself very interesting but we will not go into it here and just refer the reader to the following sections.

We now analyze the inhomogeneous term in the equation for $h_{\mu\nu}$. The tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ verifies the reduced Einstein equations of the form:

$$\widetilde{\square}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h),$$

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h), \tag{2.14}$$

$$P(\partial_\mu h, \partial_\nu h) = \frac{1}{4}\partial_\mu \mathrm{tr} h\, \partial_\nu \mathrm{tr} h - \frac{1}{2}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}, \tag{2.15}$$

Here $Q_{\mu\nu}$ are linear combinations of the standard null-forms and $G_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic form in ∂h with coefficients as smooth functions of h vanishing at $h = 0$. The weak decay estimates imply that the last two terms decay fast

$$|Q_{\mu\nu}(\partial h, \partial h)| + |G_{\mu\nu}(h)(\partial h, \partial h)| \lesssim |\overline{\partial} h|\, |\partial h| + |h||\partial h|^2 \lesssim \varepsilon^2(1 + r + t)^{-2-2\gamma}(1 + |t - r|)^{-2\gamma} \tag{2.16}$$

The problematic term is $P(\partial_\mu h, \partial_\nu h)$ since a priori the weak decay estimates only give the decay rate of $\varepsilon^2(1 + r + t)^{-1-2\gamma}(1 + |t - r|)^{-1-2\gamma}$, which is not sufficient in the wave zone $t \approx r$. The crucial improvement comes as a result of a decomposition of the tensor $P(\partial_\mu h, \partial_\nu h)$ with respect to a null frame $\{L, \underline{L}, S_1, S_2\}$. Let $T \in \mathcal{T} = \{L, S_1, S_2\}$ be any of the vectors generating the tangent space to the forward Minkowski light cones and $U \in \mathcal{U} = \{L, \underline{L}, S_1, S_2\}$ denote any of the null frame vectors. Define, for an arbitrary symmetric two tensor k, $|k|_{\mathcal{T}\mathcal{U}} = \sum_{T\in\mathcal{T}, U\in\mathcal{U}} = |T^\mu U^\mu k_{\mu\nu}|$. It then follows that

$$|P(\partial h, \partial h)|_{\mathcal{T}\mathcal{U}} \lesssim |\overline{\partial} h|\, |\partial h| \lesssim \varepsilon^2(1 + r + t)^{-2-2\gamma}(1 + |t - r|)^{-2\gamma} \tag{2.17}$$

On the other hand, the absolute value of the tensor $P(\partial h, \partial h)$ obeys the estimate

$$|P(\partial h, \partial h)| \lesssim |\partial h|^2_{\mathcal{T}\mathcal{U}} + |\partial h|_{LL}|\partial h| \tag{2.18}$$

We now decompose the system of equations for h with respect to the null-frame

$$|\widetilde{\square}_g h|_{\mathcal{T}\mathcal{U}} \lesssim \varepsilon^2(1 + r + t)^{-2-2\gamma}(1 + |t - r|)^{-2\gamma}, \tag{2.19}$$

$$|\widetilde{\square}_g h|_{\mathcal{U}\mathcal{U}} \lesssim |\partial h|^2_{\mathcal{T}\mathcal{U}} + \varepsilon^2(1 + r + t)^{-2-2\gamma}(1 + |t - r|)^{-2\gamma}, \tag{2.20}$$

where in the last inequality we also used the improved decay estimate for ∂h_{LL} obtained from the wave coordinate condition. The result is a system of equations where the components $\widetilde{\square}_g h_{TU}$ have very good decay properties, while $\widetilde{\square}_g h_{UU}$ for the remaining non tangential component depends, to the highest order, only on the components h_{TU} satisfying a better equation. An additional subtlety in the above analysis is the fact that contraction with the null frame does not commute with $\widetilde{\square}_g$ (or even with $\square$). However, the decay estimate (2.13) for the wave equation only uses the principal radial part of $\square$: $\partial_t^2 - r^{-2}\partial_r^2 - 2\, r^{-1}\partial_r$, which respects the null frame. This analysis results in the improved decay estimates

$$|\partial h|_{TU} \leq C\varepsilon(1 + t)^{-1}, \qquad |\partial h| \leq C\varepsilon(1 + t)^{-1}\ln(2 + t) \tag{2.21}$$

The energy estimates. We rely on the following energy estimate for the wave equation, which holds under the assumption that the above decay estimates hold for the background metric g: for any $\gamma > 0$

$$(2.22)\quad \int_{\Sigma_T} |\partial\phi|^2 + \int_0^T \int_{\Sigma_\tau} \frac{\gamma\, |\bar\partial\phi|^2}{(1+|t-r|)^{1+2\gamma}} \le 8\int_{\Sigma_0} |\partial\phi|^2 + C\varepsilon \int_0^T \int_{\Sigma_t} \frac{|\partial\phi|^2}{1+t} + 16\int_0^T \int_{\Sigma_t} |\widetilde{\Box}_g\phi||\partial_t\phi|$$

This implies that the energy of a solution of the homogeneous wave equation $\widetilde{\Box}_g\phi = 0$ grows but at the rate of at most $(1+t)^{C\varepsilon}$. The presence of an additional space-time integral containing tangential derivatives on the right and side of (2.22) is crucial for our analysis. This type of estimate in the constant coefficient case basically follows by averaging of the energy estimates on light cones used e.g. in [S1]. We also note that the energy estimates with space-time quantities involving special derivatives of a solution were also considered and used in the work of Alinhac, see e.g. [A2], [A3]). In our work we use the space-time integral with derivatives spanning the tangent space to outgoing light cones and weights dependent on the distance to the Minkowski light cone $r = t+1$. We emphasize that the energy estimate (2.22) is proved only under the assumption of the weak decay of all components of the background metric g together with the strong decay of the components controlled from the wave coordinate condition.

It is worth noting that a combination of the energy estimates of the type (2.22) and Klainerman-Sobolev inequalities would also yield a very simple proof of the small data global existence result for semilinear equations $\Box\phi = Q(\partial\phi, \partial\phi)$ obeying the standard null condition. This fact appears to be previously unknown.

The energy estimate (2.22) will be applied simultaneously to all components of the tensor h. As in the equations (2.19), (2.20) the inhomogeneous term obeys the following estimate:

$$|\widetilde{\Box}_g h| \lesssim \varepsilon(1+r+t)^{-3/2-\gamma}(1+|t-r|)^{1/2-\gamma}|\partial h| + \varepsilon(1+t)^{-1}|\partial h|,$$

where in the last inequality we used the improved decay estimate for the $|\partial h|_{\mathcal{TU}}$ components. The energy estimate (11.3) will then imply that

$$E_0(t) \le 16\varepsilon^2(1+t)^{C\varepsilon}.$$

Higher order energy estimates. In addition to the energy estimates for the components of the tensor h we need estimates for the higher vector field derivatives of h: $Z^I h$ with Minkowski vector fields $Z = \{\partial_\alpha, \Omega_{\alpha\beta}, S\}$. To obtain these estimates we apply Z^I to the equation $\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}$ for h. Applying vector fields to the nonlinear terms $F_{\mu\nu}$ yields similar nonlinear terms for higher derivatives and these are can be dealt with using the estimates already described above. We must note however that this is where the additional space-time integral involving the tangential derivatives on the left hand side of the energy estimate (11.3) becomes crucial. Consider for example the term $\partial h \cdot \bar\partial Z^I h$ generated by one of the null forms in $F_{\mu\nu}$. We estimate its contribution, with the help of the weak decay estimates, to the energy estimate as follows:

$$|\partial h|\,|\bar\partial Z^I h||\partial_t Z^I h| \le \frac{C\varepsilon|\partial_t Z^I h|}{(1+t)^{1/2+\gamma}} \frac{|\bar\partial Z^I h|}{(1+|t-r|)^{1/2+\gamma}} \le \frac{C\varepsilon|\partial_t Z^I h|^2}{(1+t)^{1+2\gamma}} + \frac{C\varepsilon|\bar\partial Z^I h|^2}{(1+|t-r|)^{1+2\gamma}}$$

The integral of the first term is easily controlled by the energy on time slices times an integrable factor in time. The space time integral of the second term is in fact part of the energy (2.22), and if we

choose ε sufficiently small this term can be absorbed by the space time integral on the left. The idea with the space-time integral is that one can use the extra decay in $|t-r|$ when one does not have full decay in t.

The more serious problem in higher order energy estimates lies however in the commutators between Z^I and the principal part $\widetilde{\Box}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$.

The commutators. Writing $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$ with $H^{\alpha\beta} = -m^{\alpha\alpha'}m^{\beta\beta'}h_{\alpha'\beta'} + O(h^2)$, we show the following commutator estimate[13]

$$\big|[Z, \widetilde{\Box}_g]\phi\big| \le C\Big(\frac{|H| + |ZH|}{1+t+r} + \frac{|ZH|_{LL} + |H|_{L\mathcal{T}}}{1+|\,t-r|}\Big)\sum_{|I|\le 1}|\partial Z^I\phi| \le \frac{C\varepsilon}{1+t+r}\sum_{|I|\le 1}|\partial Z^I\phi| \tag{2.23}$$

by the weak decay assumptions (2.10) and the improved decay from the wave coordinate condition (2.12). We should note that for a generic quasilinear wave equation commutators with Minkowski vector fields Z give rise to uncontrollable error terms. In the special case of the equation $\Box\phi = \phi\Delta\phi$ this problem can be overcome by modifying the vector fields Z, [A3]. In our case it is the wave coordinate gauge that provides additional cancellations.

This commutator estimate applied to $\phi = h_{\alpha\beta}$ together with the analysis in the previous section now gives estimates for the energy E_1 as well as for the stronger decay estimates for the second derivatives of h, (2.26) with $|J| = 1$. This commutator will also show up as a top order term $[Z, \widetilde{\Box}_g]\cdot Z^{I-1}h_{\alpha\beta}$ in the energy estimate for $Z^I h$ and the resulting term can be dealt with in the same way.

The other top order term generated by the commutators $[Z^I, \widetilde{\Box}_g]\phi$ is of the form $(Z^I H^{\alpha\beta})\partial_\alpha\partial_\beta$. We first apply the pointwise estimate

$$|(Z^I H^{\alpha\beta})\partial_\alpha\partial_\beta\phi| \le C\Big(\frac{|Z^I H|}{1+t+r} + \frac{|Z^I H|_{LL}}{1+|\,t-r|}\Big)\sum_{|K|\le 1}|\partial Z^K\phi|$$

To deal with its contribution to the energy estimate we use the Poincare estimate with a boundary term

$$\int_{R^3}\frac{|Z^I H|^2_{LL}\,dx}{(1+|t-r|)^{2+2\sigma}} \le C\int_{S_{(t+1)}}|Z^I H|^2_{LL}\,dS + C\int_{R^3}\frac{|\partial_r Z^I H|^2_{LL}\,dx}{(1+|t-r|)^{2\sigma}}, \qquad \sigma > -1/2, \quad \sigma \ne 1/2 \tag{2.24}$$

together with the fact that h is Schwarzschild outside the cone $r = t+1$, because of the inward bending of the Schwarzschild light cones, and hence there $|Z^I h| \le C\varepsilon/(1+t)$. The way coordinate condition implies that $|\partial Z^I H|_{LL}$ can be estimated by $|\bar\partial Z^I H|$ and lower order terms. The term involving $|\bar\partial Z^I H|$ is then controlled by the space-time integral on the left hand side.

One can use a similar but more trivial argument for decay estimates, i.e.

$$|Z^I H|_{LL} \le |Z^I H_{LL}|_{r=t+1} + (1+|t-r|)|\partial_r Z^I H_{LL}|_{L^\infty}$$

The lower order terms. So far we have only discussed the top order terms, but there will also be several lower order terms (relative to $|I| = k+1$) to deal with. These are typically of the form

$$|\partial Z^J h|\,|\partial Z^K h| \qquad \text{or} \qquad |Z^J h|\,|\partial^2 Z^{K-1}h| \le C\frac{|Z^J h|}{1+|t-r|}\,|\partial Z^K h| \tag{2.25}$$

[13] This commutator estimate applies to the vector fields $Z = \{\partial_\alpha, \Omega_{\alpha\beta}\}$. For the scaling vector field $Z = S = x^\alpha\partial_\alpha$ the commutator expression should have the form $\widetilde{\Box}_g S - (S+2)\widetilde{\Box}_g$.

with $|J|, |K| < |I| = k+1$ The lower order terms are dealt with using induction. We describe the induction argument for the decay estimates. From this it will be clear how it also proceeds for the energy estimates. We will inductively assume that we have the bounds:

$$|\partial Z^J h| + |Z^J h|(1+|t-r|)^{-1} \le C_k t^{-1+C_k\varepsilon}, \qquad |J| \le k \tag{2.26}$$

The terms in (2.25) can then be estimated by $C_k^2\varepsilon^2 t^{-2+2C_k\varepsilon}$. Including the top order terms using (2.23) applied to $\phi = Z^{I-1}h$, and using (2.13) applied to $\widetilde{\Box}_g Z^I h$ we get an inequality of the form

$$M(t) \le \int_0^t \frac{C\varepsilon\, M(s)}{1+s} + \frac{C\varepsilon^2}{(1+s)^{1-C\varepsilon}}\, ds \tag{2.27}$$

where $M(t) = (1+t)\|\partial Z^I h(t,\cdot)\|_{L^\infty}$. The Gronwall's inequality then gives the bound $M(t) \le C(1+t)^{2C\varepsilon}$.

3 The Einstein equations in wave coordinates

For a Lorentzian metric $g_{\mu\nu}$, where $\mu,\nu = 0,...,3$ we denote

$$\Gamma_{\mu\ \nu}^{\ \lambda} = \frac{1}{2} g^{\lambda\delta}\big(\partial_\mu g_{\delta\nu} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu}\big), \tag{3.1}$$

the Christoffel symbols of g and

$$R_{\mu\ \nu\delta}^{\ \lambda} = \partial_\delta \Gamma_{\mu\ \nu}^{\ \lambda} - \partial_\nu \Gamma_{\mu\ \delta}^{\ \lambda} + \Gamma_{\rho\ \delta}^{\ \lambda}\Gamma_{\mu\ \nu}^{\ \rho} - \Gamma_{\rho\ \nu}^{\ \lambda}\Gamma_{\mu\ \delta}^{\ \rho} \tag{3.2}$$

its Riemann curvature tensor with $R_{\mu\nu} = R_{\mu\ \nu\alpha}^{\ \alpha}$, the Ricci tensor.

We consider the metric g satisfying the Einstein vacuum equations

$$R_{\mu\nu} = 0. \tag{3.3}$$

We impose the wave coordinate condition:

$$\Gamma^\lambda := g^{\alpha\beta}\,\Gamma_{\alpha\ \beta}^{\ \lambda} = 0 \tag{3.4}$$

It follows that assuming (3.4) we have that the reduced wave operator $\widetilde{\Box}_g = g^{\alpha\beta}$

$$\widetilde{\Box}_g = \Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha g^{\alpha\beta}\sqrt{|g|}\partial_\beta \tag{3.5}$$

The following lemma provides the description of the Einstein vacuum equations in wave coordinates as a system of quasilinear wave equations for $g_{\mu\nu}$.

Lemma 3.1. *Let metric g satisfy the Einstein vacuum equations* (3.3) *together with the wave coordinate condition* (3.4)*. Then $g_{\mu\nu}$ solves the following system of reduced Einstein equations:*

$$\widetilde{\Box}_g g_{\mu\nu} = \widetilde{P}(\partial_\mu g, \partial_\nu g) + \widetilde{Q}_{\mu\nu}(\partial g, \partial g) \tag{3.6}$$

where

$$\widetilde{P}(\partial_\mu g, \partial_\nu g) = \frac{1}{4} g^{\alpha\alpha'} \partial_\mu g_{\alpha\alpha'} \, g^{\beta\beta'} \partial_\nu g_{\beta\beta'} - \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \partial_\mu g_{\alpha\beta} \, \partial_\nu g_{\alpha'\beta'} \tag{3.7}$$

$$\begin{aligned} \widetilde{Q}_{\mu\nu}(\partial g, \partial g) &= \partial_\alpha g_{\beta\mu} \, g^{\alpha\alpha'} g^{\beta\beta'} \partial_{\alpha'} g_{\beta'\nu} - g^{\alpha\alpha'} g^{\beta\beta'} \big(\partial_\alpha g_{\beta\mu} \, \partial_{\beta'} g_{\alpha'\nu} - \partial_{\beta'} g_{\beta\mu} \, \partial_\alpha g_{\alpha'\nu}\big) \\ &\quad + g^{\alpha\alpha'} g^{\beta\beta'} \big(\partial_\mu g_{\alpha'\beta'} \partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'} \partial_\mu g_{\beta\nu}\big) + g^{\alpha\alpha'} g^{\beta\beta'} \big(\partial_\nu g_{\alpha'\beta'} \partial_\alpha g_{\beta\mu} - \partial_\alpha g_{\alpha'\beta'} \partial_\nu g_{\beta\mu}\big) \\ &\quad + \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \big(\partial_{\beta'} g_{\alpha\alpha'} \partial_\mu g_{\beta\nu} - \partial_\mu g_{\alpha\alpha'} \partial_{\beta'} g_{\beta\nu}\big) + \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \big(\partial_{\beta'} g_{\alpha\alpha'} \partial_\nu g_{\beta\mu} - \partial_\nu g_{\alpha\alpha'} \partial_{\beta'} g_{\beta\mu}\big) \end{aligned} \tag{3.8}$$

Furthermore, the wave coordinate condition (3.4) *reads*

$$g^{\alpha\beta} \partial_\alpha g_{\beta\mu} = \frac{1}{2} g^{\alpha\beta} \partial_\mu g_{\alpha\beta}, \qquad \textit{or} \qquad \partial_\alpha g^{\alpha\nu} = \frac{1}{2} g_{\alpha\beta} g^{\nu\mu} \partial_\mu g^{\alpha\beta} \tag{3.9}$$

Proof. The proof of (3.9) is immediate.

We now observe that

$$\partial_\alpha g_{\beta\mu} = \Gamma_{\alpha\beta\mu} + \Gamma_{\alpha\mu\beta}, \qquad \text{where} \qquad \Gamma_{\mu\alpha\nu} = g_{\alpha\lambda} \Gamma_{\mu\ \nu}^{\ \lambda}.$$

It follows that $g_{\alpha\lambda} \partial_\beta \Gamma_{\mu\ \nu}^{\ \lambda} = \partial_\beta \Gamma_{\mu\alpha\nu} - (\Gamma_{\beta\alpha\lambda} + \Gamma_{\beta\lambda\alpha}) \Gamma_{\mu\ \nu}^{\ \lambda}$ so also using that $\Gamma_{\alpha\lambda\beta} = \Gamma_{\beta\lambda\alpha}$ we obtain

$$R_{\mu\alpha\nu\beta} = g_{\alpha\lambda} R_{\mu\ \nu\beta}^{\ \lambda} = \partial_\beta \Gamma_{\mu\alpha\nu} - \partial_\nu \Gamma_{\mu\alpha\beta} + \Gamma_{\nu\lambda\alpha} \Gamma_{\mu\ \beta}^{\ \lambda} - \Gamma_{\alpha\lambda\beta} \Gamma_{\mu\ \nu}^{\ \lambda} \tag{3.10}$$

It follows from (3.9) that

$$g^{\alpha\beta}\Big(\partial_\mu \partial_\alpha g_{\beta\nu} - \frac{1}{2} \partial_\mu \partial_\nu g_{\alpha\beta}\Big) = -\partial_\mu g^{\alpha\beta}\Big(\partial_\alpha g_{\beta\nu} - \frac{1}{2} \partial_\nu g_{\alpha\beta}\Big) = g^{\alpha\alpha'} g^{\beta\beta'} \, \partial_\mu g_{\alpha'\beta'} \Big(\partial_\alpha g_{\beta\nu} - \frac{1}{2} \partial_\nu g_{\alpha\beta}\Big) \tag{3.11}$$

and hence

(3.12)

$$\begin{aligned} g^{\alpha\beta}\big(\partial_\alpha \Gamma_{\mu\beta\nu} - \partial_\nu \Gamma_{\mu\beta\alpha}\big) &= \frac{g^{\alpha\beta}}{2}\big(\partial_\alpha \partial_\mu g_{\beta\nu} + \partial_\alpha \partial_\nu g_{\beta\mu} - \partial_\alpha \partial_\beta g_{\mu\nu}\big) - \frac{g^{\alpha\beta}}{2}\big(\partial_\nu \partial_\mu g_{\beta\alpha} + \partial_\nu \partial_\alpha g_{\beta\mu} - \partial_\nu \partial_\beta g_{\mu\alpha}\big) \\ &= -\frac{g^{\alpha\beta}}{2} \partial_\alpha \partial_\beta g_{\mu\nu} + \frac{g^{\alpha\beta}}{2}\big(\partial_\alpha \partial_\mu g_{\beta\nu} + \partial_\nu \partial_\beta g_{\mu\alpha} - \partial_\nu \partial_\mu g_{\beta\alpha}\big) \\ &= -\frac{1}{2} \, g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \, \big(\partial_\mu g_{\alpha'\beta'} \partial_\alpha g_{\beta\nu} + \partial_\nu g_{\alpha'\beta'} \partial_\alpha g_{\beta\mu} - \partial_\nu g_{\alpha'\beta'} \partial_\mu g_{\alpha\beta}\big). \end{aligned}$$

Here by (3.9) we can write

$$\begin{aligned} g^{\alpha\alpha'} g^{\beta\beta'} \, \partial_\mu g_{\alpha'\beta'} \partial_\alpha g_{\beta\nu} &= g^{\alpha\alpha'} g^{\beta\beta'} \, \partial_\alpha g_{\alpha'\beta'} \partial_\mu g_{\beta\nu} + g^{\alpha\alpha'} g^{\beta\beta'} \, \big(\partial_\mu g_{\alpha'\beta'} \partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'} \partial_\mu g_{\beta\nu}\big) \\ &= \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \, \partial_{\beta'} g_{\alpha'\alpha} \, \partial_\mu g_{\beta\nu} + g^{\alpha\alpha'} g^{\beta\beta'} \, \big(\partial_\mu g_{\alpha'\beta'} \, \partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'} \, \partial_\mu g_{\beta\nu}\big) \\ &= \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \, \partial_\mu g_{\alpha'\alpha} \, \partial_{\beta'} g_{\beta\nu} + g^{\alpha\alpha'} g^{\beta\beta'} \Big(\frac{1}{2}\big(\partial_{\beta'} g_{\alpha'\alpha} \, \partial_\mu g_{\beta\nu} - \partial_\mu g_{\alpha'\alpha} \, \partial_{\beta'} g_{\beta\nu}\big) + \big(\partial_\mu g_{\alpha'\beta'} \, \partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'} \, \partial_\mu g_{\beta\nu}\big)\Big) \\ &= \frac{1}{4} g^{\alpha\alpha'} g^{\beta\beta'} \, \partial_\mu g_{\alpha'\alpha} \, \partial_\nu g_{\beta\beta'} + g^{\alpha\alpha'} g^{\beta\beta'} \Big(\frac{1}{2}\big(\partial_{\beta'} g_{\alpha'\alpha} \, \partial_\mu g_{\beta\nu} - \partial_\mu g_{\alpha'\alpha} \, \partial_{\beta'} g_{\beta\nu}\big) + \big(\partial_\mu g_{\alpha'\beta'} \, \partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'} \, \partial_\mu g_{\beta\nu}\big)\Big) \end{aligned} \tag{3.13}$$

Hence by (3.13) and (3.13) with μ and ν interchanged we get

(3.14)

$$\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\Big(\partial_\mu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\nu}+\partial_\nu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\mu}-\partial_\nu g_{\alpha'\beta'}\,\partial_\mu g_{\alpha\beta}\Big)=g^{\alpha\alpha'}g^{\beta\beta'}\,\big(\frac{1}{4}\partial_\mu g_{\alpha'\alpha}\,\partial_\nu g_{\beta\beta'}-\frac{1}{2}\partial_\nu g_{\alpha'\beta'}\,\partial_\mu g_{\alpha\beta}\big)$$
$$+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\Big(\big(\partial_\mu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\nu}-\partial_\alpha g_{\alpha'\beta'}\,\partial_\mu g_{\beta\nu}\big)+\big(\partial_\nu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\mu}-\partial_\alpha g_{\alpha'\beta'}\,\partial_\nu g_{\beta\mu}\big)\Big)$$
$$\frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}\Big(\big(\partial_{\beta'}g_{\alpha'\alpha}\,\partial_\mu g_{\beta\nu}-\partial_\mu g_{\alpha'\alpha}\,\partial_{\beta'}g_{\beta\nu}\big)+\big(\partial_{\beta'}g_{\alpha'\alpha}\,\partial_\nu g_{\beta\mu}-\partial_\nu g_{\alpha'\alpha}\,\partial_{\beta'}g_{\beta\mu}\big)\Big)$$

On the other hand

$$\text{(3.15)}\quad \Gamma_{\nu\alpha\beta}\Gamma_\mu{}^{\alpha\beta}=\frac{1}{4}\big(\partial_\nu g_{\beta\alpha}+\partial_\beta g_{\alpha\nu}-\partial_\alpha g_{\beta\nu}\big)g^{\alpha\alpha'}g^{\beta\beta'}\big(\partial_\mu g_{\beta'\alpha'}+\partial_{\beta'}g_{\alpha'\mu}-\partial_{\alpha'}g_{\beta'\mu}\big)$$
$$=\frac{1}{4}\partial_\nu g_{\alpha\beta}\;g^{\alpha\alpha'}g^{\beta\beta'}\,\partial_\mu g_{\alpha'\beta'}+\frac{1}{2}\partial_\alpha g_{\beta\mu}\;g^{\alpha\alpha'}g^{\beta\beta'}\partial_{\alpha'}g_{\beta'\nu}-\frac{1}{2}\partial_\alpha g_{\beta\mu}\;g^{\alpha\alpha'}g^{\beta\beta'}\partial_{\beta'}g_{\alpha'\nu}$$
$$=g^{\alpha\alpha'}g^{\beta\beta'}\,\Big(\frac{1}{4}\partial_\nu g_{\alpha\beta}\;\partial_\mu g_{\alpha'\beta'}+\frac{1}{2}\partial_\alpha g_{\beta\mu}\;\partial_{\alpha'}g_{\beta'\nu}-\frac{1}{2}\partial_{\beta'}g_{\beta\mu}\;\partial_\alpha g_{\alpha'\nu}\Big)$$
$$-\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\,\Big(\partial_\alpha g_{\beta\mu}\;\partial_{\beta'}g_{\alpha'\nu}-\partial_{\beta'}g_{\beta\mu}\;\partial_\alpha g_{\alpha'\nu}\Big)$$
$$=g^{\alpha\alpha'}g^{\beta\beta'}\,\Big(\frac{1}{4}\partial_\nu g_{\alpha\beta}\;\partial_\mu g_{\alpha'\beta'}-\frac{1}{8}\partial_\mu g_{\beta\beta'}\;\partial_\nu g_{\alpha\alpha'}+\frac{1}{2}\partial_\alpha g_{\beta\mu}\;\partial_{\alpha'}g_{\beta'\nu}\Big)$$
$$-\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\,\Big(\partial_\alpha g_{\beta\mu}\;\partial_{\beta'}g_{\alpha'\nu}-\partial_{\beta'}g_{\beta\mu}\;\partial_\alpha g_{\alpha'\nu}\Big)$$

where the last inequality follows from (3.9).

Taking the trace of (3.10) and using (3.12), (3.4) we obtain

$$\text{(3.16)}\;\; R_{\mu\nu}=-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}+\Gamma_{\nu\alpha\beta}\Gamma_\mu{}^{\alpha\beta}+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\,\Big(\partial_\mu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\nu}+\partial_\nu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\mu}-\partial_\nu g_{\alpha'\beta'}\,\partial_\mu g_{\alpha\beta}\Big),$$

Using (3.15) and (3.14) we get

$$\text{(3.17)}\quad R_{\mu\nu}=-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}+g^{\alpha\alpha'}g^{\beta\beta'}\,\Big(-\frac{1}{4}\partial_\nu g_{\alpha\beta}\;\partial_\mu g_{\alpha'\beta'}+\frac{1}{8}\partial_\mu g_{\beta\beta'}\;\partial_\nu g_{\alpha\alpha'}\Big)$$
$$+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\partial_\alpha g_{\beta\mu}\;\partial_{\alpha'}g_{\beta'\nu}-\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\,\Big(\partial_\alpha g_{\beta\mu}\;\partial_{\beta'}g_{\alpha'\nu}-\partial_{\beta'}g_{\beta\mu}\;\partial_\alpha g_{\alpha'\nu}\Big)$$
$$+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\Big(\big(\partial_\mu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\nu}-\partial_\alpha g_{\alpha'\beta'}\,\partial_\mu g_{\beta\nu}\big)+\big(\partial_\nu g_{\alpha'\beta'}\,\partial_\alpha g_{\beta\mu}-\partial_\alpha g_{\alpha'\beta'}\,\partial_\nu g_{\beta\mu}\big)\Big)$$
$$\frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}\Big(\big(\partial_{\beta'}g_{\alpha'\alpha}\,\partial_\mu g_{\beta\nu}-\partial_\mu g_{\alpha'\alpha}\,\partial_{\beta'}g_{\beta\nu}\big)+\big(\partial_{\beta'}g_{\alpha'\alpha}\,\partial_\nu g_{\beta\mu}-\partial_\nu g_{\alpha'\alpha}\,\partial_{\beta'}g_{\beta\mu}\big)\Big)$$

The result now follows. □

Let m denote the standard Minkowski metric

$$m_{00}=-1,\qquad m_{ii}=1,\quad\text{if}\quad i=1,...,3,\quad\text{and}\quad m_{\mu\nu}=0,\quad\text{if}\quad \mu\neq\nu,$$

Define a 2-tensor h from the decomposition

$$g_{\mu\nu}=m_{\mu\nu}+h_{\mu\nu}.$$

Let $m^{\mu\nu}$ be the inverse of $m_{\mu\nu}$. Then for small h

$$H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu} = -h^{\mu\nu} + O^{\mu\nu}(h^2), \qquad \text{where} \qquad h^{\mu\nu} = m^{\mu\mu'} m^{\nu\nu'} h_{\mu'\nu'}$$

and $O^{\mu\nu}(h^2)$ vanishes to second order at $h = 0$.

As a consequence of Lemma 3.1 we get:

Lemma 3.2. *If Einstein's equation's* (3.3) *and the wave coordinate condition* (3.4) *hold then*

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) \tag{3.18}$$

where $F_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic form in ∂h with coefficients that are smooth functions of h. More precisely,

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h) \tag{3.19}$$

where

$$P(\partial_\mu h, \partial_\nu h) = \frac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'} \, m^{\beta\beta'} \partial_\nu h_{\beta\beta'} - \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta} \, \partial_\nu h_{\alpha'\beta'} \tag{3.20}$$

and

$$\begin{aligned} Q_{\mu\nu}(\partial h, \partial h) &= \partial_\alpha h_{\beta\mu} \, m^{\alpha\alpha'} m^{\beta\beta'} \partial_{\alpha'} h_{\beta'\nu} - m^{\alpha\alpha'} m^{\beta\beta'} \big(\partial_\alpha h_{\beta\mu} \, \partial_{\beta'} h_{\alpha'\nu} - \partial_{\beta'} h_{\beta\mu} \, \partial_\alpha h_{\alpha'\nu}\big) \\ &\quad + m^{\alpha\alpha'} m^{\beta\beta'} \big(\partial_\mu h_{\alpha'\beta'} \, \partial_\alpha h_{\beta\nu} - \partial_\alpha h_{\alpha'\beta'} \, \partial_\mu h_{\beta\nu}\big) + m^{\alpha\alpha'} m^{\beta\beta'} \big(\partial_\nu h_{\alpha'\beta'} \, \partial_\alpha g h_{\beta\mu} - \partial_\alpha h_{\alpha'\beta'} \, \partial_\nu h_{\beta\mu}\big) \\ &\quad + \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \big(\partial_{\beta'} h_{\alpha\alpha'} \, \partial_\mu h_{\beta\nu} - \partial_\mu h_{\alpha\alpha'} \, \partial_{\beta'} h_{\beta\nu}\big) + \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \big(\partial_{\beta'} h_{\alpha\alpha'} \, \partial_\nu h_{\beta\mu} - \partial_\nu h_{\alpha\alpha'} \, \partial_{\beta'} h_{\beta\mu}\big) \end{aligned}$$

is a null form and $G_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic form in ∂h with coefficients smoothly dependent on h and vanishing when h vanishes: $G_{\mu\nu}(0)(\partial h, \partial h) = 0$.

Furthermore

$$m^{\alpha\beta} \partial_\alpha h_{\beta\mu} = \frac{1}{2} m^{\alpha\beta} \partial_\mu h_{\alpha\beta} + G_\mu(h)(\partial h), \qquad \text{or} \qquad \partial_\alpha H^{\alpha\nu} = \frac{1}{2} g_{\alpha\beta} \big(m^{\nu\mu} + H^{\nu\mu}\big) \partial_\mu H^{\alpha\beta} \tag{3.21}$$

where $G_\mu(h)(\partial h)$ is a linear function of ∂h with coefficients that are smooth functions of h and that vanishes when h vanishes: $G_\mu(0)(\partial h) = 0$.

Observe that the terms in (3.20) do not satisfy the classical null condition. However the trace $m^{\mu\nu} h_{\mu\nu}$ satisfies a nonlinear wave equation with semilinear terms obeying the the null condition:

$$g^{\alpha\beta} \partial_\alpha \partial_\beta m^{\mu\nu} h_{\mu\nu} = Q(\partial h, \partial h) + G(h)(\partial h, \partial h).$$

4 The initial data

In this section we discuss the initial data for which the results of our paper apply. We shall consider the asymptotically flat data, satisfying a global smallness condition, with the property that it coincides with the Schwarzschild data outside the ball of radius one.

We start by showing the existence of such data. Let (g_0, k_0) be asymptotically flat initial data for the Einstein equations consisting of the Riemannian metric g_0 and a second fundamental form k_0. The initial data for the vacuum Einstein satisfy the constraint equations

$$R_0 - (\mathrm{tr} k_0)^2 + |k_0|^2 = 0, \tag{4.1}$$

$$\nabla^j k_{0ij} - \nabla_i \mathrm{tr} k_0 = 0 \tag{4.2}$$

We restrict our attention to the time-symmetric case $R_0 = k_0 = 0$. Then, if (g_0, k_0) is sufficiently close to the Minkowski data and g_0 satisfies the parity condition $g_0(x) = g_0(-x)$, by the results of Corvino [Co] and Chrusciel-Delay [C-D] one can construct a new set of initial data (g, k) with the properties that

- The initial data (g, k) coincides with (g_0, k_0) on the ball of radius 1/2.
- (g, k) is exactly the Schwarzschild data $(g_S^x, 0)$ of mass M outside B_1, the ball of radius one.

At this point we specify the smallness conditions:

$$M \le \epsilon, \qquad \sum_{0 \le |I| \le N} \Big(\|\partial_x^I (g - \delta)\|_{L^2(B_1)} + \sum_{0 \le |J| \le N-1} \|\partial_x^J k\|_{L^2(B_1)} \Big) \le \epsilon \tag{4.3}$$

for some sufficiently large integer N. Here ∂_x^I denotes the derivative $\partial_{x_1}^{I_1} ... \partial_{x_n}^{I_n}$, where $(I_1, .., I_n)$ is an arbitrary multi-index with the property that $I_1 + .. + I_n = |I|$.

We have two expressions for the Schwarzschild metric in isotropic and wave coordinates:

$$g_S = -\frac{(1 - M/r)^2}{(1 + M/r)^2} dt^2 + (1 + \frac{M}{r})^4 dx^2, \tag{4.4}$$

$$g_s = -\frac{r - 2M}{r + 2M} dt^2 + \frac{r + 2M}{r - 2M} dr^2 + (r + 2M)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{4.5}$$

The expressions g_S^x and g_s^x will denote the spatial parts the Schwarzschild metric in respective coordinates. Observe that

$$g_s = m + \frac{4M}{r}(dt^2 + dx^2) + O(r^{-2}), \tag{4.6}$$

We now find the coordinate change transforming the metric g_S into g_s. Set

$$t = \tau, \qquad r = \rho + \frac{M^2}{\rho} \tag{4.7}$$

In the coordinates τ, ρ the metric g_s takes the form g_S. This change of coordinates is one-to-one for the values $\rho > M$. Since the mass $M << 1$ we can define the change of coordinates $r = \Phi(\rho)$, where Φ coincides with the map (4.7) for $\rho > 1$ and the identity transformation for $\rho \le 1/2$. Thus we have constructed the initial data (g, k) such that

- The initial data (g, k) coincides (in new coordinates) with (g_0, k_0) on the ball of radius 1/2.
- (g, k) is exactly the Schwarzschild data $(g_s^x, 0)$ outside the ball of radius one.

- Moreover, the new data still obeys the smallness condition (4.3).

The constructed metric is already in wave coordinates on its Schwarzschild part. We now describe the procedure which produces the initial data $(g, \partial_t g)$ associated with (g, k) and satisfying the wave coordinate condition.

Recall that *a priori* we are only given the spatial part of the metric g_{ij} together with a second fundamental form k_{ij}. We now define the full space-time metric $g_{\alpha\beta}$ on the Cauchy hypersurface Σ_0 as follows:

$$g_{0i} = 0, \qquad g_{00} = -a(r), \tag{4.8}$$

where the function

$$a(r) = \frac{r - 2M}{r + 2M}, \qquad \text{for } r > 1$$
$$a(r) = 1, \qquad \text{for } r \le \frac{1}{2}$$

Thus defined metric coincides with the full Schwarzschild metric g_s for $r > 1$. We further define

$$\partial_t g_{ij} = -2a k_{ij} \tag{4.9}$$

It remains to determine $\partial_t g_{0\alpha}$. We find it by satisfying the wave coordinate condition

$$g^{\beta\mu}\partial_\mu g_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\partial_\alpha g_{\mu\nu}$$

Setting $\alpha = 0$ we obtain

$$\frac{1}{2} g^{00}\partial_t g_{00} = -g^{\beta i}\partial_i g_{0\beta} + \frac{1}{2} g^{ij}\partial_t g_{ij}$$

This defines $\partial_t g_{00}$. On the other hand setting $\alpha = i$ we obtain

$$g^{00}\partial_t g_{0i} = -g^{\beta j}\partial_j g_{i\beta} + \frac{1}{2} g^{\mu\nu}\partial_i g_{\mu\nu}$$

This determines $\partial_t g_{0i}$. Observe that since the metric g coincides with the Schwarzschild metric g_s, already satisfying the wave coordinate condition, outside the ball of radius one, we have that on that set the initial data takes the form $(g_s, 0)$. Hence we constructed the initial data $(g, \partial_t g)$ with the properties that

- The initial data $(g, \partial_t g)$ corresponds to the initial data (g, k) prescribed originally.
- $(g, \partial_t g)$ is exactly the Schwarzschild data $(g_s, 0)$ outside the ball of radius one.
- The initial data verifies the wave coordinate condition.
- The initial data satisfies the smallness condition

$$\sum_{0 \le |I| \le N} \Big(\|\partial_x^I (g - m)\|_{L^2(B_1)} + \sum_{0 \le |J| \le N-1} \|\partial_x^J \partial_t g\|_{L^2(B_1)} \Big) \le \epsilon \tag{4.10}$$

Now with the initial data $(g, \partial_t g)$ we solve the reduced Einstein equations (3.6). It follows from the proof of Lemma 3.1 that, in the notation $\Gamma^\lambda = g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}$, the reduced Einstein equations can be written in the form:

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}(D_\alpha \Gamma_\beta + D_\beta \Gamma_\alpha) - \Gamma_\sigma N^\sigma_{\alpha\beta}(g, \partial g) = 0. \tag{4.11}$$

Here D denotes a covariant derivative with respect to the space-time metric g and $N^\sigma_{\alpha\beta}$ are some given functions depending on g and ∂g. Observe that the initial data $(g, \partial_t g)$ were chosen in such a way that the wave coordinate condition $\Gamma^\lambda = 0$ is satisfied on the initial hypersurface Σ_0. We now argue that this condition is propagated, i.e, the solution of the reduced Einstein equations (4.11) obeys $\Gamma^\lambda = 0$ on *any* hypersuface Σ_t. We would have thus shown that a solution of the reduced Einstein equations is, in fact, a solution of the vacuum Einstein equations.

To prove that $\Gamma^\lambda = 0$ we differentiate (4.11) and use the contracted Bianchi identity $D^\beta \mathbf{R}_{\alpha\beta} = \frac{1}{2} D_\alpha \mathbf{R}$

$$\begin{aligned} 0 = 2(D^\beta \mathbf{R}_{\alpha\beta} - \frac{1}{2} D_\alpha \mathbf{R}) &= D^\beta D_\alpha \Gamma_\beta + D^\beta D_\beta \Gamma_\alpha - D_\alpha D^\beta \Gamma_\beta - 2D^\beta(\Gamma_\sigma N^\sigma_{\alpha\beta}) - D_\alpha(\Gamma_\sigma N^{\sigma\beta}_\beta) \\ &= D^\beta D_\beta \Gamma_\alpha + \mathbf{R}_{\alpha\gamma}\Gamma^\gamma - 2(D_\beta \Gamma_\sigma) N^{\sigma\beta}_\beta - (D_\alpha \Gamma_\sigma) N^{\sigma\beta}_\beta - 2\Gamma_\sigma (D_\beta N^{\sigma\beta}_\alpha) - \Gamma_\sigma (D_\alpha N^{\sigma\beta}_\beta) \end{aligned}$$

Therefore, Γ^λ satisfies a covariant wave equation, on the background determined by the constructed metric g, with the initial condition $\Gamma^\lambda = 0$. It remains to show that $D_t \Gamma^\lambda = 0$ on Σ_0 and the conclusion that $\Gamma^\lambda \equiv 0$ will follows by the uniqueness result for wave equation.

We recall that the initial data (g, k) verifies the constraint equations (4.1), (4.2), which imply that on Σ_0

$$\mathbf{R}_{TT} + \frac{1}{2}\mathbf{R} = 0, \qquad \mathbf{R}_{Ti} = 0,$$

where $T = -(g_{00})^{-1}\partial_t$ is the unit future oriented normal to Σ_0. Therefore returning to (4.11) we obtain that

$$\begin{aligned} 0 &= \mathbf{R}_{00} + \frac{1}{2}\mathbf{R} = -(g_{00})^{-1} D_t \Gamma_0 + D^i \Gamma_i, \\ 0 &= \mathbf{R}_{0i} = \frac{1}{2} D_t \Gamma_i + \frac{1}{2} D_i \Gamma_0 \end{aligned}$$

This finishes the proof that $\Gamma^\lambda \equiv 0$.

We also know that the time-independent Schwarzschild metric g_s is a solution of the Einstein vacuum equation $\mathbf{R}_{\alpha\beta} = 0$. Moreover, since g_s satisfies the wave coordinate condition it also verifies the reduced Einstein equations (4.11). Since the initial data $(g, \partial_t g) = (g_s, 0)$ outside the ball of radius two, constructed solution will coincide with the Schwarzschild solution in the exterior of the null cone developed from the sphere of radius one in Σ_0.

We end the discussion of the initial data by comparing the light cones of Minkowski and Schwarzschild spaces in the wave coordinates of the Schwarzschild space.

Lemma 4.1. *For an arbitrary $R > 2M$ the forward null cone of the metric g_s, intersecting the time slice $t = 0$ along the sphere of radius R, is contained in the interior of the Minkowski cone $t - r = R$.*

Proof. The null cone intersecting the time slice $t = 0$ along the sphere of radius R can be realized as the level hypersurface $u = 0$ of the optical function u solving the eikonal equation

$$g_s^{\alpha\beta}\partial_\alpha u\ \partial_\beta u = 0$$

with the initial condition that $u = 0$ on the sphere of radius R at time $t = 0$. Because of the spherical symmetry of the Schwarzschild metric g_s and the initial condition we look for a spherically symmetric solution $u = u(t, r)$. The eikonal equation then reads

$$\frac{r+2M}{r-2M}(\partial_t u)^2 = \frac{r-2M}{r+2M}(\partial_r u)^2$$

Let $t = \gamma(r)$ be a null geodesic, originating from some point on the sphere of radius R at $t = 0$, such that $u(\gamma(r), r) = 0$. Then

$$\partial_t u\dot{\gamma}(r) + \partial_r u = 0$$

Substituting this into the eikonal equation we obtain that

$$\Big(\frac{r+2M}{r-2M}\Big)^2 = |\dot{\gamma}(r)|^2$$

Taking the square root and integrating we obtain that

$$\gamma(r) = \gamma(R) \pm \Big(r - R + 4M\ln\frac{r-2M}{R-2M}\Big)$$

Thus the null geodesics are described the curves

$$t = \pm\Big(r - R + 4M\ln\frac{r-2M}{R-2M}\Big)$$

In particular, the forward null cone is contained in the interior of the set $t \ge r - R$. $\square$

5 The null-frame and null-forms

Below we introduce a standard Minkowski null-frame used throughout the paper. At each point (t, x) we introduce a pair of null vectors $(L, \underline{L})$

$$L^0 = 1, \quad L^i = x^i/|x|, \quad i = 1,2,3, \quad \text{and} \quad \underline{L}^0 = 1, \quad \underline{L}^i = -x^i/|x|, \quad i = 1,2,3,$$

Adding two orthonormal tangent to the sphere S^2 vectors S_1, S_2 which are orthogonal to ω defines a null frame $(L, \underline{L}, S_1, S_2)$.

Remark 5.1. Since S^2 does not admit a global orthonormal frame S_1, S_2 we could alternatively introduce a global frame induced by the projections of the coordinate vector fields e_i.

Let P be the orthogonal projection of a vector field in $\mathbf{R}^3$ along $\omega = x/|x|$ onto the tangent space of the sphere; $PV = V - \langle V, \omega\rangle\omega$. For $i = 1,2,3$ denote the projection of ∂_i by

$$\text{(5.1)} \qquad \not{\partial}_i = A_i^j\partial_j = \partial_i - \omega_i\omega^j\partial_j, \qquad \text{where} \qquad A_i^j = (Pe_i)^j = \delta_i^j - \omega_i\omega^j, \qquad i = 1,2,3,$$

where e_i is the usual orthonormal basis in $\mathbf{R}^3$, and the sums are over $j = 1,2,3$ only. Let $\bar{\partial}_0 = L^\alpha\partial_\alpha$ and $\bar{\partial}_i = \not{\partial}_i$, for $i = 1,2,3$. Then a linear combination of the derivatives $\{\bar{\partial}_0, ..., \bar{\partial}_3\}$ spans the tangent space of the forward light cone.

In what follows A, B will denote any of the vectorfields S_1, S_2. We will use the summation conventions

$$X^A A^\alpha = X^\beta S_{1\beta} S_1^\alpha + X^\beta S_{2\beta} S_2^\alpha, \qquad X_A Y_A = X^\alpha Y^\beta S_{1\alpha} S_{1\beta} + X^\alpha Y^\beta S_{2\alpha} S_{2\beta}.$$

Obvious generalizations of the above conventions will be used for higher order tensors.

We record the following null frame decomposition of a vector field $X = X^\alpha \partial_\alpha$: $X^\alpha = X^L L^\alpha + X^{\underline{L}} \underline{L}^\alpha + X^A A^\alpha$. Relative to a null frame the Minkowski metric m has the following form

$$m_{LL} = m_{\underline{L}\underline{L}} = m_{LA} = m_{\underline{L}A} = 0, \qquad m_{L\underline{L}} = m_{\underline{L}L} = -2, \qquad m_{AB} = \delta_{AB},$$

i.e. $m_{\alpha\beta} X^\alpha Y^\beta = -2(X^L Y^{\underline{L}} + X^{\underline{L}} Y^L) + X^A Y^A$. Recall that we raise and lower indices of any tensor relative to the Minkowski metric m, i.e., $X_\alpha = m_{\alpha\beta} X^\beta$. We define $X_Y = m_{\alpha\beta} X^\alpha Y^\beta = X_\alpha Y^\alpha$. Then $X_Y = X^L Y_L + X^{\underline{L}} Y_{\underline{L}} + X^A Y_A$. It is useful to remember the following rule:

$$X^L = -\frac{1}{2} X_{\underline{L}}, \quad X^{\underline{L}} = -\frac{1}{2} X_L, \quad X^A = X_A.$$

Then

$$m^{LL} = m^{\underline{L}\underline{L}} = m^{LA} = m^{\underline{L}A} = 0, \qquad m^{L\underline{L}} = m^{\underline{L}L} = -1/2, \qquad m^{AB} = \delta^{AB}$$

i.e. $m^{\alpha\beta} X_\alpha Y_\beta = -\frac{1}{2}\big(X_L Y_{\underline{L}} + X_{\underline{L}} Y_L\big) + X_A Y_A$.

Definition 5.2. *Denote $q = r - t$ and $s = t + r$ the null coordinates of the Minkowski metric m and $\partial_q = \frac{1}{2}(\partial_r - \partial_t)$ and $\partial_s = \frac{1}{2}(\partial_t + \partial_r)$, the corresponding null vector fields*

Let $k_{XY} = k_{\alpha\beta} X^\alpha Y^\beta$. Then

$$\text{tr } k = m^{\alpha\beta} k_{\alpha\beta} = -\frac{1}{2}\big(k_{L\underline{L}} + k_{\underline{L}L}\big) + \overline{\text{tr}}\, k \tag{5.2}$$

where

$$\overline{\text{tr}}\, k = \delta^{AB} k_{AB} = \overline{\delta}^{ij} k_{ij}, \qquad \text{and} \qquad \overline{\delta}^{ij} = \delta^{ij} - \omega^i \omega^j \tag{5.3}$$

where the sum is over $i, j = 1, 2, 3$ only.

If k and p are symmetric it follows that

$$\begin{aligned} p_{\alpha\beta} k^{\alpha\beta} &= m^{\alpha\alpha'} m^{\beta\beta'} p_{\alpha\beta} k_{\alpha'\beta'} \\ &= \frac{1}{4}\big(p_{LL} k_{\underline{L}\underline{L}} + p_{\underline{L}\underline{L}} k_{LL} + 2 p_{L\underline{L}} k_{L\underline{L}}\big) - \delta^{AB}\big(p_{AL} k_{B\underline{L}} + p_{A\underline{L}} k_{BL}\big) + \delta^{AB} \delta^{A'B'} p_{AA'} k_{BB'} \\ &= \frac{1}{4}\big(p_{LL} k_{\underline{L}\underline{L}} + p_{\underline{L}\underline{L}} k_{LL} + 2 p_{L\underline{L}} k_{\underline{L}L}\big) - \overline{\delta}^{ij}\big(p_{iL} k_{j\underline{L}} + p_{i\underline{L}} k_{jL}\big) + \overline{\delta}^{ij} \overline{\delta}^{i'j'} p_{ii'} k_{jj'} \end{aligned} \tag{5.4}$$

Lemma 5.3. *With $P(p,k)$ given by (3.20) we have for symmetric 2-tensors p and k:*

$$\begin{aligned} P(p,k) &= \frac{1}{4} m^{\alpha\beta} p_{\alpha\beta} m^{\alpha\beta} k_{\alpha\beta} - \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} p_{\alpha\beta} k_{\alpha'\beta'} \\ &= -\frac{1}{8}\big(p_{LL} k_{\underline{L}\underline{L}} + p_{\underline{L}\underline{L}} k_{LL}\big) - \frac{1}{4} \delta^{AB} \delta^{A'B'} \Big(2 p_{AA'} k_{BB'} - p_{AB} k_{A'B'}\Big) \\ &\qquad + \frac{1}{4} \delta^{AB}\big(2 p_{AL} k_{B\underline{L}} + 2 p_{A\underline{L}} k_{BL} - p_{AB} k_{L\underline{L}} - p_{L\underline{L}} k_{AB}\big) \end{aligned} \tag{5.5}$$

i.e. at least one of the factors contains only tangential components.

Furthermore

$$p^{\alpha\beta}\partial_\alpha = p^{L\beta}\partial_L + p^{\underline{L}\beta}\partial_{\underline{L}} + p^{A\beta}\partial_A = -\frac{1}{2}p_{\underline{L}}{}^{\beta}\partial_L - \frac{1}{2}p_L{}^{\beta}\partial_{\underline{L}} + p^{A\beta}\partial_A$$

We introduce the following notation. Let $\mathcal{T} = \{L, S_1, S_2\}$, $\mathcal{U} = \{\underline{L}, L, S_1, S_2\}$, $\mathcal{L} = \{L\}$ and $\mathcal{S} = \{S_1, S_2\}$. For any two of these families $\mathcal{V}$ and $\mathcal{W}$ and an arbitrary two-tensor p we denote

$$|p|_{\mathcal{V}\mathcal{W}} = \sum_{V\in\mathcal{V}, W\in\mathcal{W},} |p_{\beta\gamma}V^\beta W^\gamma|, \tag{5.6}$$

$$|\partial p|_{\mathcal{V}\mathcal{W}} = \sum_{U\in\mathcal{U}, V\in\mathcal{V}, W\in\mathcal{W},} |(\partial p)_{\alpha\beta\gamma}U^\alpha V^\beta W^\gamma|, \tag{5.7}$$

$$|\overline{\partial} p|_{\mathcal{V}\mathcal{W}} = \sum_{T\in\mathcal{T}, V\in\mathcal{V}, W\in\mathcal{W},} |(\partial p)_{\alpha\beta\gamma}T^\alpha V^\beta W^\gamma| \tag{5.8}$$

Let Q denote a null form, i.e. $Q_{\alpha\beta}(\partial\phi,\partial\psi) = \partial_\alpha\phi\,\partial_\beta\psi - \partial_\beta\phi\,\partial_\alpha\psi$ if $\alpha\neq\beta$ and $Q_0(\partial\phi,\partial\psi) = m^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\psi$.

Lemma 5.4. *If P is as in Lemma 5.3 then*

$$|P(p,k)| \lesssim |p\,|_{\mathcal{T}\mathcal{U}}|k|_{\mathcal{T}\mathcal{U}} + |p\,|_{LL}|k| + |p\,|\,|k|_{LL} \tag{5.9}$$

If $Q(\partial\phi,\partial\psi)$ is a null form then

$$|Q(\partial\phi,\partial\psi)| \lesssim |\overline{\partial}\phi||\partial\phi| + |\partial\phi||\overline{\partial}\psi| \tag{5.10}$$

Furthermore

$$|k^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi| \lesssim \big(|k|_{LL}|\partial\phi|^2 + |k|\,|\bar{\partial}\phi||\partial\phi|\big) \tag{5.11}$$

$$|L_\alpha k^{\alpha\beta}\partial_\beta\phi| \lesssim \big(|k|_{LL}|\partial\phi| + |k|\,|\bar{\partial}\phi|\big) \tag{5.12}$$

$$|(\partial_\alpha k^{\alpha\beta})\partial_\beta\phi| \lesssim \big(|\partial k|_{LL} + |\overline{\partial}k|\big)\,|\partial\phi| + |\partial k|\,|\overline{\partial}\phi| \tag{5.13}$$

Proof. The proof of (5.10) for the null form Q_0 follows directly from (5.2). To prove the claim for the null forms $Q_{\alpha\beta}$ use that

$$\partial_i = L_i(\partial_s + \partial_q) + \bar{\partial}_i, \quad i = 1,2,3, \qquad \partial_0 = L_0(\partial_s - \partial_q) \tag{5.14}$$

Therefore,

$$|Q_{\alpha\beta}(\partial\phi,\partial\psi)| = |\partial_\alpha\phi\partial_\beta\psi - \partial_\beta\phi\partial_\alpha\psi| \leq C|\bar{\partial}\phi|\,|\partial\psi| + C|\partial\phi|\,|\bar{\partial}\psi|$$

The estimates (5.11)-(5.13) follow from (5.4). ∎

Lemma 5.5. *If $k^{\alpha\beta}$ is a symmetric tensor and ϕ a function then*

$$|k^{\alpha\beta}\partial_\alpha\partial_\beta\phi| \lesssim \big(|k|_{LL}|\partial^2\phi| + |k|\,|\bar\partial\partial\phi|\big) \tag{5.15}$$

Also, with $\overline{\operatorname{tr}}\, k = \delta^{AB}k_{AB} = (\delta^{ij} - \omega^i\omega^j)k_{ij}$ *we have*

$$\big|k^{\alpha\beta}\partial_\alpha\partial_\beta\phi - k_{LL}\partial_q^2\phi - 2k_{L\underline{L}}\partial_s\partial_q\phi - r^{-1}\,\overline{\operatorname{tr}}\, k\,\partial_q\phi\big| \lesssim |k|_{L\mathcal{T}}|\bar\partial\partial\phi| + |k|\,\big(|\bar\partial^2\phi| + r^{-1}|\bar\partial\phi|\big). \tag{5.16}$$

Proof. The estimate (5.15) follow from (5.4). We have

$$\partial_i\omega_j = r^{-1}(\delta_{ij} - \omega_i\omega_j) = r^{-1}\overline{\delta}_{ij} \tag{5.17}$$

Furthermore $\partial_i = \bar\partial_i + \omega_i\partial_r$, where $\partial_r = \omega^j\partial_j$ so $[\bar\partial_i, \partial_r] = (\bar\partial_i\omega^k)\partial_k$ and

$$\begin{aligned}\partial_i\partial_j &= (\bar\partial_i + \omega_i\partial_r)(\bar\partial_j + \omega_j\partial_r)\\ &= \bar\partial_i\bar\partial_j + \omega_i\,\omega^k\,\bar\partial_j\partial_k + \omega_j\omega^k\,\bar\partial_i\partial_k + \omega_i\omega_j\partial_r^2 + (\bar\partial_i\omega_j)\partial_r + \omega_j(\bar\partial_i\omega^k)\partial_k\\ &= \bar\partial_i\bar\partial_j + \omega_i\,\bar\partial_j\partial_r + \omega_j\,\bar\partial_i\partial_r + \omega_i\omega_j\partial_r^2 + r^{-1}\overline{\delta}_{ij}\partial_r - r^{-1}\omega_i\bar\partial_j\end{aligned} \tag{5.18}$$

Furthermore

$$\partial_0\partial_i = \partial_t(\bar\partial_i + \omega_i\partial_r) = \bar\partial_i\partial_t + \omega_i\partial_t\partial_r \tag{5.19}$$

Hence

$$\begin{aligned}k^{\alpha\beta}\partial_\alpha\partial_\beta &= k^{00}\partial_t^2 + 2k^{0i}\omega_i\partial_t\partial_r + k^{ij}\omega_i\omega_j\partial_r^2 + r^{-1}\overline{\operatorname{tr}}\,k\,\partial_r\\ &\quad + k^{ij}\bar\partial_i\bar\partial_j - r^{-1}k^{ij}\omega_i\bar\partial_j + 2k^{0j}\bar\partial_j\partial_t + 2k^{ij}\omega_i\bar\partial_j\partial_r\end{aligned} \tag{5.20}$$

If we substitute $\partial_t = \partial_s - \partial_q$, $\partial_r = \partial_s + \partial_q$ and identify

$$k_{LL} = k^{00} - 2k^{0i}\omega_i + k^{ij}\omega_i\omega_j, \qquad k_{L\underline{L}} = -k^{00} + k^{ij}\omega_i\omega_j, \qquad k_{\underline{L}\underline{L}} = k^{00} + 2k^{0i}\omega_i + k^{ij}\omega_i\omega_j \tag{5.21}$$

and

$$-k^{0j} + k^{ij}\omega_j = k_0^{\ j} + k_i^{\ j}\omega^i = k_L^{\ j}, \qquad k^{0j} + k^{ij}\omega_j = -k_0^{\ j} + k_i^{\ j}\omega^i = k_{\underline{L}}^{\ j} \tag{5.22}$$

we get

$$\begin{aligned}k^{\alpha\beta}\partial_\alpha\partial_\beta &= k_{LL}\partial_q^2 + 2k_{L\underline{L}}\partial_s\partial_q + k_{\underline{L}\underline{L}}\partial_s^2 + r^{-1}\overline{\operatorname{tr}}\,k\,\partial_q\\ &\quad + k^{ij}\bar\partial_i\bar\partial_j + r^{-1}\overline{\operatorname{tr}}\,k\,\partial_s - r^{-1}k^{ij}\omega_i\bar\partial_j + 2k_L^{\ j}\bar\partial_j\partial_q + 2k_{\underline{L}}^{\ j}\bar\partial_j\partial_s\end{aligned} \tag{5.23}$$

Finally, we can also write

$$2k_L^{\ j}\bar\partial_j\partial_q = k_L^{\ j}\bar\partial_j(\omega^k\partial_k - \partial_t) = k_L^{\ j}\omega^k\bar\partial_j\partial_k - k_L^{\ j}\bar\partial_j\partial_t + r^{-1}k_L^{\ j}\bar\partial_j, \tag{5.24}$$

since $(\bar\partial_j\omega^k)\partial_k = r^{-1}\bar\partial_j$. The inequality (5.16) now follows. □

Corollary 5.6. *Let ϕ be a solution of the reduced wave equation $\widetilde{\Box}_g\phi = F$ with a metric g such that $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ satisfies the condition that $|H^{L\underline{L}}| < \frac{1}{4}$. Then*

(5.25)
$$\Big|\Big(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q - \frac{\overline{\operatorname{tr}}\, H + H_{L\underline{L}}}{2g^{L\underline{L}}\ r}\Big)\partial_q(r\phi) + \frac{rF}{2g^{L\underline{L}}}\Big| \lesssim r|\triangle_\omega\phi| + |H|_{L\mathcal{T}}\ r\,|\bar\partial\partial\phi| + |H|\,\big(r\,|\bar\partial^2\phi| + |\bar\partial\phi| + r^{-1}|\phi|\big)$$

where $\triangle_\omega = \bar\triangle = \delta^{ij}\bar\partial_i\bar\partial_j$.

Proof. Define the new metric
$$\tilde g^{\alpha\beta} = \frac{g^{\alpha\beta}}{-2g^{L\underline{L}}}.$$
The equation $g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = F$ then takes the form
$$\tilde g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = \frac{F}{-2g^{L\underline{L}}},$$
which can also be written as
$$\Box\phi + (\tilde g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\partial_\beta\phi = \frac{F}{-2g^{L\underline{L}}}$$
Let $k^{\alpha\beta}$ be the tensor $k^{\alpha\beta} = (\tilde g^{\alpha\beta} - m^{\alpha\beta})$ Observe that
$$\begin{aligned} k^{\alpha\beta} = (-2g^{L\underline{L}})^{-1}\big(g^{\alpha\beta} + 2m^{\alpha\beta}g^{L\underline{L}}\big) &= (-2g^{L\underline{L}})^{-1}\big(H^{\alpha\beta} + m^{\alpha\beta}(2g^{L\underline{L}} + 1)\big)\\ &= (-2g^{L\underline{L}})^{-1}\big(H^{\alpha\beta} + 2m^{\alpha\beta}H^{L\underline{L}}\big)\end{aligned}$$

Thus,

(5.26)
$$k_{L\underline{L}} = 0, \qquad k_{L\mathcal{T}} = (-2g^{L\underline{L}})^{-1}H_{L\mathcal{T}}, \qquad \overline{\operatorname{tr}}\, k\ = (-2g^{L\underline{L}})^{-1}\big(\overline{\operatorname{tr}}\, H\ + H_{L\underline{L}}\big)$$

Moreover, $|k| \lesssim |H|$, since $g^{L\underline{L}} = H^{L\underline{L}} - \frac{1}{2}$ and by the assumptions of the Corollary $|H^{L\underline{L}}| < \frac{1}{4}$.

Now using (5.16) of Lemma 5.5, with the condition that $k_{L\underline{L}} = 0$, together with the decomposition
$$\Box\phi = -\partial_t^2\phi + \triangle\phi = \frac{1}{r}(\partial_t + \partial_r)(\partial_r - \partial_t)r\phi + \triangle_\omega\phi = \frac{4}{r}\partial_s\partial_q r\phi + \triangle_\omega\phi.$$
we find that the identity $\Box\phi + k^{\alpha\beta}\partial_\alpha\partial_\beta\phi = (-2g^{L\underline{L}})^{-1}F$ leads to the inequality
$$\big|4\partial_s\partial_q r\phi + rk_{LL}\partial_q^2\phi + \overline{\operatorname{tr}}\, k\,\partial_q\phi + (2g^{L\underline{L}})^{-1}rF\big| \lesssim r|\triangle_\omega\phi| + r|k|_{L\mathcal{T}}|\bar\partial\partial\phi| + |k|\,\big(r\,|\bar\partial^2\phi| + |\bar\partial\phi|\big)$$
Finally, identity (5.26) and a crude estimate $|k| \lesssim |H|$ yield the desired result. ◻

6 The weak null condition and asymptotic expansion of Einstein's equation's in wave coordinates

Let us now first describe the weak null condition. The results of this section appear in [L-R]. Consider the Cauchy problem for a system of nonlinear wave equations in three space dimensions:

(6.1)
$$-\Box\, u_i\ = F_i(u, u', u''), \qquad i = 1, ..., N, \qquad u = (u_1, ..., u_N),$$

where $-\Box = -\partial_t^2 + \sum_{j=1}^3 \partial_{x^j}^2$. We assume that F is a function of u and its derivatives of the form

$$F_i(u,u',u'') = a_{i\alpha\beta}^{jk} \partial^\alpha u_j\, \partial^\beta u_k + G_i(u,u',u''), \tag{6.2}$$

where $G_i(u,u',u'')$ vanishes to third order as $(u,u',u'')\to 0$ and $a_{i\,\alpha\beta}^{jk}=0$ unless $|\alpha|\le|\beta|\le 2$ and $|\beta|\ge 1$. Here we used the summation convention over repeated indices. We assume that the initial data

$$u(0,x) = \varepsilon u_0(x) \in C^\infty, \quad u_t(0,x) = \varepsilon u_1(x) \in C^\infty \tag{6.3}$$

is small and decays fast as $|x|\to\infty$. We are going to determine conditions on the nonlinearity such that the equation (6.1) is compatible with the asymptotic expansion as $|x|\to\infty$ and $|x|\sim t$

$$u(t,x) \sim \varepsilon U(q,s,\omega)/|x|, \qquad \text{where} \quad q = |x|-t,\ \ s = \varepsilon \ln|x|,\ \ \omega = x/|x|, \tag{6.4}$$

for all sufficiently small $\epsilon > 0$. The linear and some nonlinear wave equations allow for such an expansion with U independent of s and the next term decaying like $\varepsilon/|x|^2$, see [H1, H2]. Substituting (6.4) into (6.1) and equating powers of order $\varepsilon^2/|x|^2$ we see that

$$2\partial_s\partial_q U_i = A_{i\,mn}^{jk}(\omega)(\partial_q^m U_j)(\partial_q^n U_k), \qquad U\big|_{s=0} = F_0, \tag{6.5}$$

where

$$A_{i,mn}^{jk}(\omega) = \sum_{|\alpha|=m,|\beta|=n} a_{i,\alpha\beta}^{jk}\hat\omega^\alpha\hat\omega^\beta, \qquad \text{where } \hat\omega = (-1,\omega) \quad \text{and } \hat\omega^\alpha = \hat\omega_{\alpha_1}...\hat\omega_{\alpha_k} \tag{6.6}$$

In fact, $\Box u = -\varepsilon^{-1}\partial_s\partial_q(ru) +$ angular derivatives and $\partial_\mu = \hat\omega_\mu\partial_q +$ tangential derivatives.

One can show that (6.1)-(6.3) has a solution as long as $\varepsilon\log t$ is bounded, provided that $\varepsilon > 0$ is sufficiently small and the solution of (6.5) exists up to that time, [J-K, H1, H2, L1, L2]. The only exception is the case $A_{i00}^{jk} \neq 0$, which has shorter life span. In cases where the solution of (6.5) blows up it has been shown that solutions of (6.1)-(6.3) also break down in some finite time $T_\varepsilon \le e^{C/\varepsilon}$, [J1, H1, A1]. John's example was

$$\Box u = u_t \triangle u \tag{6.7}$$

for which (6.5) is the Burger's equation $(2\partial_s - U_q\partial_q)U_q = 0$, which is known to blow up. The equation

$$\Box u = u_t^2 \tag{6.8}$$

is another example where solutions blow up, for which (6.5) is $\partial_s U_q = U_q^2$, that also blows up.

The *null condition* of [K2] is equivalent to

$$A_{i\,mn}^{jk}(\omega) = 0 \quad \text{for all } (i,j,k,m,n),\ \omega\in\mathbb{S}^2. \tag{6.9}$$

The results of [C1], [K2] assert that (6.1)-(6.3) has global solutions for all sufficiently small initial data, provided that the null condition is satisfies. In this case the asymptotic equation (6.5) trivially can be solved globally. Moreover, similar to the linear case, its solutions approach a limit as $s\to\infty$ and

the solutions of (6.1)-(6.3) decay like solutions of linear equations. A typical example of an equation satisfying the null condition is

$$\Box u = u_t^2 - |\nabla_x u|^2 \tag{6.10}$$

There is however a more general class of nonlinearities for which solutions of (6.5) do not blow up:

We say that a system (6.1) satisfies the *weak null condition* if the solutions of the corresponding asymptotic system (6.5) exist for all s and if the solutions together with its derivatives grow at most exponentially in s for all initial data decaying sufficiently fast in q.

Under the weak null condition assumption solutions of (6.5) satisfy the equation (6.1) up to terms of order $\varepsilon^2/|x|^{3-C\varepsilon}$, but need only decay like $\varepsilon/|x|^{1-C\varepsilon}$. An example of the equation satisfying the weak null condition is given by

$$\Box u = u \triangle u \tag{6.11}$$

In [L2] it was proven that (6.11) have small global solutions in the spherically symmetric case and recently [A3] established this result without the symmetry assumption. The equation (6.11) appears to be similar to (6.7) but a closer look shows that the corresponding asymptotic equation:

$$(2\partial_s - U\partial_q)U_q = 0 \tag{6.12}$$

has global solutions growing exponentially in s, see [L2]. The system

$$\Box u = v_t^2, \qquad \Box v = 0 \tag{6.13}$$

is another example that satisfies the weak null condition. The equation (6.13) appears to resemble (6.8). The system however decouples: v satisfies a linear homogeneous equation and given v we have a linear inhomogeneous equation for u, and global existence follows. The corresponding asymptotic system is

$$\partial_s\partial_q U = (\partial_q V)^2, \qquad \partial_s\partial_q V = 0 \tag{6.14}$$

The solution of the second equation in (6.14) is independent of s: $V_q = V_q(q,\omega)$ and substituting this into the first equation we see that $U_q(s,q,\omega) = sV_q(q,\omega)^2$ so ∂u only decays like $|x|^{-1}\ln|x|$.

We show below that the Einstein vacuum equations in wave coordinates satisfy the weak null condition, i.e. that the corresponding asymptotic system (6.5) admits global solutions. In fact, each of the quadratic nonlinear terms in the Einstein equations is either of the type appearing in (6.10), (6.11) or (6.13).

Theorem 6.1. *Let h be a symmetric 2-tensor and let*

$$h_{\mu\nu}(t,x) \sim \varepsilon U_{\mu\nu}(s,q,\omega)/|x|, \qquad \text{where} \qquad q = |x| - t,\ \ s = \varepsilon \ln|x|,\ \ \omega = x/|x|. \tag{6.15}$$

be an asymptotic ansatz. Then the asymptotic system for the the Einstein equations in wave coordinates (3.18), obtained by formally equating the terms with the coefficients $\epsilon^2|x|^{-2}$, takes the following form:

$$\big(2\partial_s - U_{LL}\partial_q\big)\partial_q U_{\mu\nu} = L_\mu L_\nu P(\partial_q U, \partial_q U), \qquad \forall \mu,\nu = 0,...,3 \tag{6.16}$$

where $U_{LL} = m^{\alpha\alpha'} m^{\beta\beta'} U_{\alpha'\beta'} L_\alpha L_\beta$ and $P(\partial_q U, \partial_q U) = \frac{1}{4}\partial_q trU\, \partial_q trU - \frac{1}{2}\partial_q U_{\alpha\beta}\, \partial_q U^{\alpha\beta}$. The asymptotic form of the wave coordinate condition (3.21) *is*

$$2\partial_q U_{L\mu} = L_\mu\, \partial_q\, \mathrm{tr}\, U, \qquad \forall \mu = 0, ..., 3, \tag{6.17}$$

where $U_{L\mu} = m^{\alpha\alpha'} U_{\alpha'\mu} L_\alpha$ and $trU = m^{\alpha\beta} U_{\alpha\beta}$ The solution of the system (6.16)-(6.17) *exists globally and, thus, the Einstein vacuum equations* (3.18) *in wave coordinates satisfies the weak null condition. Moreover, the component $\partial_q U_{\underline{L}\underline{L}}$ grows at most as s while the remaining components are uniformly bounded.*

The asymptotic form (6.16) follows by a direct calculation from (3.18). Observe that the null form $Q_{\mu\nu}(\partial h, \partial h)$ disappears after passage to the asymptotic system.

Next we note that (6.17) is preserved under the flow of (6.16). Contracting (6.16) with $L^\mu L^\nu$ we obtain

$$(2\partial_s - U_{LL}\partial_q)\partial_q U_{LL} = 0,$$

which can be solved globally. More generally, contracting (6.16) with the vector fields $\{L, S_1, S_2\}$ we obtain

$$(2\partial_s - U_{LL}\partial_q)\partial_q U_{TU} = 0, \qquad \text{if} \quad T \in \{L, S_1, S_2\} \quad \text{and} \quad U \in \{L, \underline{L}, S_1, S_2\}, \tag{6.18}$$

which can be solved globally now that U_{LL} has been determined. Note that the components $\partial_q U_{TU}$ are constant along the integral curves of the vector field $2\partial_s - U_{LL}\partial_q$. The remaining unknown component $U_{\underline{L}\underline{L}}$ can be determined by contracting the equation (6.16) with the vector field $\underline{L}$.

$$(2\partial_s - U_{LL}\partial_q)\partial_q U_{\underline{L}\underline{L}} = 4P(\partial_q U, \partial_q U), \tag{6.19}$$

By Lemma 5.3 the quantity $P(\partial_q U, \partial_q U)$ does not contain the term $(\partial_q U_{\underline{L}\underline{L}})^2$. Thus, the equation (6.19) can be solved globally and produces solutions growing exponentially in s. A more precise information can be obtained from the asymptotic form of the wave coordinate condition (6.17). For contracting it with the null frame $\{L, S_1, S_2\}$ we obtain $\partial_q U_{LT} = 0$, if $T \in \{L, S_1, S_2\}$. Therefore,

$$P(\partial_q U, \partial_q U) = -\frac{1}{4}\delta^{AB}\delta^{A'B'}\Big(2\partial_q U_{AA'}\, \partial_q U_{BB'} - \partial_q U_{AB}\, \partial_q U_{A'B'}\Big) - \frac{1}{2}\delta^{AB}\partial_q U_{AB}\, \partial_q U_{L\underline{L}}, \tag{6.20}$$

It follows from (6.18) that P is already determined and is, in fact, constant along the characteristics of the field $2\partial_s - U_{LL}\partial_q$. Therefore, integrating (6.19) we infer that $\partial_q U_{\underline{L}\underline{L}}$ grows at most like s.

7 Vector fields and commutators

Let $Z \in \mathcal{Z}$ be any of the vector fields

$$\Omega_{\alpha\beta} = -x_\alpha \partial_\beta + x_\beta \partial_\alpha, \qquad S = t\partial_t + r\partial_r, \qquad \partial_\alpha,$$

where $x_0 = -t$ and $x_i = x^i$, for $i \geq 1$. Let $I = (\iota_1, ..., \iota_k)$, where $|\iota_i| = 1$, be an ordered multiindex of length $|I| = k$ and let $Z^I = Z^{\iota_1} \cdots Z^{\iota_k}$ denote a product of $|I|$ such derivatives. With a slight abuse of notation we will also identify the index set with vector fields, so $I = Z$ means the index I corresponding to the vector field Z. Furthermore, by a sum over $I_1 + I_2 = I$ we mean a sum over all

possible order preserving partitions of the ordered multiindex I into two ordered multiindices I_1 and I_2, i.e. if $I=(\iota_1,...,\iota_k)$, then $I_1=(\iota_{i_1},...,\iota_{i_n})$ and $I_2=(\iota_{i_{n+1}},...,\iota_{i_k})$, where $i_1,...,i_k$ is any reordering of the integers $1,...,k$ such that $i_1<...<i_n$ and $i_{n+1}<...<i_k$ and $i_1,...,i_k$. With this convention Leibniz rule becomes $Z^I(fg)=\sum_{I_1+I_2=I}(Z^{I_1}f)(Z^{I_2}g)$. We denote by $\bar\partial$ the tangential derivatives, i.e., $\bar\partial=\{\bar\partial_0,\bar\partial_1,\bar\partial_2,\bar\partial_3\}$ and note that the span of the tangential derivatives $\{\bar\partial_0,\bar\partial_1,\bar\partial_2,\bar\partial_3\}$ coincides with the linear span of the vectorfields $\{\partial_s,\partial_{S_1},\partial_{S_2}\}$.

Lemma 7.1. *We have the following expressions for the coordinate vector fields:*

$$\partial_t=\frac{tS-x^i\Omega_{0i}}{t^2-r^2},\tag{7.1}$$

$$\partial_r=\omega^i\partial_i=\frac{t\omega^i\Omega_{0i}-rS}{t^2-r^2},\tag{7.2}$$

$$\partial_i=\frac{-x^j\Omega_{ij}+t\Omega_{0i}-x_iS}{t^2-r^2}=-\frac{x_iS}{t^2-r^2}+\frac{x_ix^j\Omega_{0j}}{t(t^2-r^2)}+\frac{\Omega_{0i}}{t}\tag{7.3}$$

In particular,

$$\partial_s=\frac12\big(\partial_t+\partial_r\big)=\frac{S+\omega^i\Omega_{0i}}{2(t+r)},\qquad \bar\partial_i=\partial_i-\omega_i\partial_r=\frac{\omega^j\Omega_{ij}}{r}=\frac{-\omega_i\omega^j\Omega_{0j}+\Omega_{0i}}{t}.\tag{7.4}$$

Lemma 7.2. *For any function f we have the estimate*

$$(1+t+|q|)|\bar\partial f|+(1+|q|)|\partial f|\lesssim C\sum_{|I|=1}|Z^If|,\qquad |\partial f|\lesssim|\bar\partial f|+|\partial_q f|\tag{7.5}$$

where $|\bar\partial f|^2=|\bar\partial_0 f|^2+|\bar\partial_1 f|^2+|\bar\partial_2 f|^2+|\bar\partial_3 f|^2$ and $\bar\partial_0=\partial_s$. Furthermore

$$|\bar\partial^2 f|\lesssim\frac{C}{r}\sum_{|I|\le 2}\frac{|Z^If|}{1+t+|q|},\tag{7.6}$$

where $|\bar\partial^2 f|^2=\sum_{\alpha,\beta=0,1,2,3}|\bar\partial_\alpha\bar\partial_\beta f|^2$.

Moreover, if $k^{\alpha\beta}$ is a symmetric tensor then

$$|k^{\alpha\beta}\partial_\alpha\partial_\beta\phi|\le C\Big(\frac{|k|}{1+t+|q|}+\frac{|k|_{LL}}{1+|q|}\Big)\sum_{|I|\le1}|\partial Z^I\phi|\tag{7.7}$$

Proof. First we note that if $r+t\le1$ then (7.5) holds since the usual derivatives ∂_α are included in the sum on the right. The inequality for $|\bar\partial f|$ in (7.5) follows directly from (7.4); one has to divide into two cases $r\le t$ and $r\ge t$ and use two different expressions depending on the relative size of r and t. The inequality for $|\partial f|$ in (7.5) follows from (7.1) and the first identity in (7.3).

If $t+r<1$ then (7.6) follows from (7.4), since $|\partial_i\omega_j|\le Cr^{-1}$ and the sum on the right of (7.6) contains the usual derivatives. Since $|\Omega_{ij}\omega_k|\le C$ and $\Omega_{ij}r=\Omega_{ij}t=0$, for $1\le i,j\le3$ it follows, by applying $\bar\partial_i=r^{-1}\omega^j\Omega_{ij}$ to the expressions in (7.4), that

$$|\bar\partial_i\bar\partial_\alpha f|\le Cr^{-1}(t+r)^{-1}\sum_{|I|\le2}|Z^If|.\tag{7.8}$$

Once again we distinguish the cases $r < t$ and $r > t$ and use different expressions for $\bar{\partial}_i$. With the notation $\bar{\partial}_0 = 2\partial_s$ (7.8) holds also for $\alpha = 0$. Since $[\partial_s, \bar{\partial}_i] = 0$ it only remains to prove (7.6) for ∂_s^2. Since $S\omega^j = 0$, $|\Omega_{0i}\omega^j| \le Ctr^{-1}$, $S(t+r) = 2(t+r)$ and $|\Omega_{0i}(t+r)| \le C(t+r)$,(7.6) follows also for ∂_s^2.

The inequality (7.7) follows from Lemma 5.5, (7.5) and the commutator identity $[Z, \partial_i] = a_i^j \partial_j$. $\square$

Lemma 7.3. *Suppose* $\widetilde{\square}_g \phi = F$. *Then*

$$\Big|\Big(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q - \frac{\overline{\mathrm{tr}}\, H\, + H_{L\underline{L}}}{2g^{L\underline{L}}\ r}\Big)\partial_q(r\phi) + \frac{rF}{2g^{L\underline{L}}}\Big| \lesssim \Big(1 + \frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|} + |H|\Big) r^{-1} \sum_{|I|\le 2} |Z^I \phi| \tag{7.9}$$

Proof. By Corollary 5.6

$$\Big|\Big(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q - \frac{\overline{\mathrm{tr}}\, H\, + H_{L\underline{L}}}{2g^{L\underline{L}}\ r}\Big)\partial_q(r\phi) + \frac{rF}{2g^{L\underline{L}}}\Big|$$
$$\lesssim r|\triangle_\omega \phi| + r|H|_{L\mathcal{T}}|\bar{\partial}\partial\phi| + |H|\,\big(r\,|\bar{\partial}^2\phi| + |\bar{\partial}\phi| + r^{-1}|\phi|\big)$$

where $\triangle_\omega = \delta^{ij}\bar{\partial}_i\bar{\partial}_j$. Here all the the derivatives can be reexpressed in terms of the vector fields Z and ∂_q using 7.2, yielding the expression (7.9). Note that

$$|\bar{\partial}\partial\phi| \lesssim \frac{\sum_{|I|=1}|Z^I\partial\phi|}{1+t+|q|} \lesssim \frac{\sum_{|I|\le 1}|\partial Z^I\phi|}{1+t+|q|} \lesssim \frac{\sum_{|I|\le 2}|Z^I\phi|}{(1+|q|)(1+t+|q|)}.$$

$\square$

Lemma 7.4. *Let* $Z = Z^\mu\partial_\mu$ *be any of the vector fields above and let* c_α^μ *be defined by*

$$[\partial_\alpha, Z] = c_\alpha{}^\mu\partial_\mu, \qquad c_\alpha{}^\mu = \partial_\alpha Z^\mu$$

Then c_α^μ *are constants and*

$$c_{LL} = c^{\underline{L}\underline{L}} = 0.$$

Furthermore

$$[Z, \square] = -c_Z\square$$

where c_Z *is either* 0 *or* 2.

In addition, if Q *is a null form, then*

$$ZQ(\partial\phi, \partial\psi) = Q(\partial\phi, \partial Z\psi) + Q(\partial Z\phi, \partial\psi) + \tilde{Q}(\partial\phi, \partial\psi) \tag{7.10}$$

for some null form $\tilde{Q}$ *on the right hand-side.*

Proof. Since $Z = Z^\alpha\partial_\alpha$ is a Killing or conformally Killing vector field we have

$$\partial_\alpha Z_\beta + \partial_\beta Z_\alpha = f m_{\alpha\beta} \tag{7.11}$$

where $Z_\alpha = m_{\alpha\beta}Z^\beta$. In fact, for the vector fields above, $f = 0$ unless $Z = S$ in which case $f = 2$. In particular,

$$L^\alpha L^\beta \partial_\alpha Z_\beta = 0.$$

If ${c_\alpha}^\mu$ is as defined above and $c_{\alpha\beta} = {c_\alpha}^\mu m_{\mu\beta} = \partial_\alpha Z_\beta$ the above simply means that $c_{LL} = c^{\underline{LL}} = 0$. which proves the first part of the lemma. To verify (7.10) we first consider the null form $Q = Q_{\alpha\beta}$ We have

$$
\begin{aligned}
ZQ_{\alpha\beta}(\partial\phi, \partial\psi) &= Q_{\alpha\beta}(\partial Z\phi, \partial\psi) + Q_{\alpha\beta}(\partial\phi, \partial Z\psi) \\
&\quad + [Z, \partial_\alpha]\phi\partial_\beta\psi - \partial_\beta\phi[Z, \partial_\alpha]\psi + [Z, \partial_\beta]\phi\partial_\alpha\psi - \partial_\alpha\phi[Z, \partial_\beta]\psi \\
&= Q_{\alpha\beta}(\partial Z\phi, \partial\psi) + Q_{\alpha\beta}(\partial\phi, \partial Z\psi) - c^\mu_\alpha(\partial_\mu\phi\partial_\beta\psi - \partial_\beta\phi\partial_\mu\psi) - c^\mu_\beta(\partial_\mu\phi\partial_\alpha\psi - \partial_\alpha\phi\partial_\mu\psi) \\
&= Q_{\alpha\beta}(\partial Z\phi, \partial\psi) + Q_{\alpha\beta}(\partial\phi, \partial Z\psi) - c^\mu_\alpha Q_{\mu\beta}(\partial\phi, \partial\psi) - c^\mu_\beta Q_{\mu\alpha}(\partial\phi, \partial\psi)
\end{aligned}
$$

The calculation for the null form $Q_0(\partial\phi, \partial\psi) = m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$ proceeds as follows:

$$
\begin{aligned}
ZQ_0(\partial\phi, \partial\psi) &= Q_0(\partial Z\phi, \partial\psi) + Q_0(\partial\phi, \partial Z\psi) + m^{\alpha\beta}[Z, \partial_\alpha]\phi\partial_\beta\psi + m^{\alpha\beta}\partial_\alpha\phi[Z, \partial_\beta]\psi \\
&= Q_0(\partial Z\phi, \partial\psi) + Q_0(\partial\phi, \partial Z\psi) + m^{\alpha\beta}c^\mu_\alpha\partial_\mu\phi\partial_\beta\psi + m^{\alpha\beta}c^\mu_b\partial_\alpha\phi\partial_\mu\psi \\
&= Q_0(\partial Z\phi, \partial\psi) + Q_0(\partial\phi, \partial Z\psi) + fm^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi \\
&= Q_0(\partial Z\phi, \partial\psi) + Q_0(\partial\phi, \partial Z\psi) + fQ_0(\partial\phi, \partial\psi),
\end{aligned}
$$

where f is a constant associated with a Killing (conf. Killing) vector field Z via a relation $c^{\alpha\beta} + c^{\beta\alpha} = fm^{\alpha\beta}$. □

Lemma 7.5. *If $k^{\alpha\beta}$ is a symmetric tensor then*

$$
k^{\alpha\beta}[\partial_\alpha\partial_\beta, Z] = k^{\alpha\beta}_Z\partial_\alpha\partial_\beta, \qquad \textit{where} \qquad k^{\alpha\beta}_Z = k^{\alpha\gamma}{c_\gamma}^\beta + k^{\gamma\beta}{c_\gamma}^\alpha, \qquad {c_\alpha}^\mu = \partial_\alpha Z^\mu. \tag{7.12}
$$

In particular $k^{\alpha\beta}_S = 2k^{\alpha\beta}$ and

$$
|k_Z|_{LL} \le 2|k|_{L\mathcal{T}}. \tag{7.13}
$$

In general

$$
[k^{\alpha\beta}\partial_\alpha\partial_\beta, Z^I] = \sum_{I_1+I_2=I,\,|I_2|<|I|} k^{I_1\alpha\beta}\partial_\alpha\partial_\beta Z^{I_2}, \tag{7.14}
$$

where

$$
k^{J\alpha\beta} = \sum_{|K|\le|J|} c^{J\alpha\beta}_{K\mu\nu}Z^K k^{\mu\nu} = -Z^J k^{\alpha\beta} - \sum_{K+Z=J} Z^K k^{\alpha\beta}_Z + \sum_{|K|\le|J|-2} d^{J\alpha\beta}_{K\mu\nu}Z^K k^{\mu\nu} \tag{7.15}
$$

for some constants $c^{J\alpha\beta}_{M\mu\nu}$ and $d^{J\alpha\beta}_{M\mu\nu}$. Here the sum (7.14) means the sum over all possible order preserving partitions of the ordered multiindex I into two ordered multiindices I_1 and I_2.

Proof. First observe that since the vector fields Z are linear in t and x we have

$$
[\partial^2_{\alpha\beta}, Z] = [\partial_\beta, Z]\partial_\alpha + [\partial_\alpha, Z]\partial_\beta = {c_\beta}^\gamma\partial_\gamma\partial_\alpha + {c_\alpha}^\gamma\partial_\gamma\partial_\beta.
$$

which proves the first statement and the second follows since $c^{\underline{L}}_L = 0$.

To prove (7.14) we first write

$$Z^I\big(k^{\alpha\beta}\partial_\alpha\partial_\beta\phi\big) = \sum_{K+J=I} (Z^K k^{\alpha\beta}) Z^J\big(\partial_\alpha\partial_\beta\phi\big)$$

Then we observe that

$$Z^J\partial_\alpha\partial_\beta\phi = \sum_{J_1+J_2=J,\, J_1=(\iota_{11},\dots,\iota_{1n})} \big[Z^{\iota_{11}}, \big[Z^{\iota_{12}}, \big[\dots, \big[Z^{\iota_{1\,n-1}}, [Z^{\iota_{1n}}, \partial^2_{\alpha\beta}]\big]\dots\big]\big]\big]\, Z^{J_2}\phi, \tag{7.16}$$

where the sum is over all order preserving partitions of the ordered multiindex $J = (\iota_1, \dots, \iota_k)$ into two ordered multiindices $J_1 = (\iota_{11}, \dots, \iota_{1n})$ and $J_2 = (\iota_{21}, \dots, \iota_{2k})$. It therefore follows that

$$k^{J\alpha\beta} = -\sum_{K+L=J,\, L=(\iota_1,\dots,\iota_l)} (Z^K k^{\alpha\beta})\, \big[Z^{\iota_1}, \big[Z^{\iota_2}, \big[\dots, \big[Z^{\iota_{l-1}}, [Z^{\iota_l}, \partial^2_{\alpha\beta}]\big]\dots\big]\big]\big]$$

The desired representation follows after taking into account that

$$(Z^K k^{\alpha\beta})[Z, \partial^2_{\alpha\beta}] = -(Z^K k_Z^{\alpha\beta})\partial_\alpha\partial_\beta$$

□

Corollary 7.6. *Let* $\widetilde{\Box}_g = \Box + H^{\alpha\beta}\partial_\alpha\partial_\beta$. *Then with* $\hat{Z} = Z + c_Z$

$$\widetilde{\Box}_g Z\phi - \hat{Z}\widetilde{\Box}_g\phi = -(\hat{Z}H^{\alpha\beta} + H_Z^{\alpha\beta})\partial_\alpha\partial_\beta\phi, \tag{7.17}$$

As a consequence, we have

$$\big|\widetilde{\Box}_g Z\phi - \hat{Z}\widetilde{\Box}_g\phi\big| \lesssim \bigg(\frac{|ZH| + |H|}{1+t+|q|} + \frac{|ZH|_{LL} + |H|_{L\mathcal{T}}}{1+|q|}\bigg)\sum_{|I|\le 1} |\partial Z^I\phi| \tag{7.18}$$

In general

$$\widetilde{\Box}_g Z^I\phi - \hat{Z}^I\widetilde{\Box}_g\phi = -\sum_{I_1+I_2=I,\, |I_2|<|I|} \hat{H}^{I_1\alpha\beta}\partial_\alpha\partial_\beta Z^{I_2}\phi, \tag{7.19}$$

where

$$\hat{H}^{J\alpha\beta} = \sum_{|M|\le|J|} c^{J\alpha\beta}_{M\mu\nu}\hat{Z}^M H^{\mu\nu} = -\hat{Z}^J H^{\alpha\beta} - \sum_{M+Z=J}\hat{Z}^M H_Z^{\alpha\beta} + \sum_{|M|\le|J|-2} d^{J\alpha\beta}_{M\mu\nu}\hat{Z}^M H^{\mu\nu} \tag{7.20}$$

We have

$$|\widetilde{\Box}_g Z^I\phi| \lesssim |\hat{Z}^I\widetilde{\Box}_g\phi| + \frac{1}{1+t+|q|}\sum_{|K|\le|I|,}\ \sum_{|J|+(|K|-1)_+\le|I|} |Z^J H|\, |\partial Z^K\phi| \tag{7.21}$$

$$+\frac{1}{1+|q|}\sum_{|K|\le|I|}\Big(\sum_{|J|+(|K|-1)_+\le|I|} |Z^J H|_{LL} + \sum_{|J'|+(|K|-1)_+\le|I|-1} |Z^{J'}H|_{L\mathcal{T}} + \sum_{|J''|+(|K|-1)_+\le|I|-2} |Z^{J''}H|\Big)|\partial Z^K\phi|$$

where $(|K|-1)_+ = |K|-1$ *if* $|K|\ge 1$ *and* $(|K|-1)_+ = 0$ *if* $|K| = 0$.

Proof. First observe that

$$\begin{aligned}\hat{Z}\widetilde{\Box}_g\phi &= (Z+c_Z)\Box\phi + (Z+c_Z)H^{\alpha\beta}\partial^2_{\alpha\beta}\phi\\ &= \Box Z\phi + H^{\alpha\beta}\partial^2_{\alpha\beta}Z\phi + (ZH^{\alpha\beta})\partial^2_{\alpha\beta}\phi + (H_Z^{\alpha\beta} + c_Z H^{\alpha\beta})\partial^2_{\alpha\beta}\phi\\ &= \widetilde{\Box}_g Z\phi + (ZH^{\alpha\beta})\partial^2_{\alpha\beta}\phi + (H_Z^{\alpha\beta} + c_Z H^{\alpha\beta})\partial^2_{\alpha\beta}\phi\end{aligned}$$

Recall now that the constant c_Z is different from 0 only in the case of the scaling vector field S. Moreover, in that case

$$H_S^{\alpha\beta} + c_S H^{\alpha\beta} = 0$$

The inequality (7.18) now follows from (7.17), (7.13) and the estimate (7.7). The general commutation formula (7.19) follows from the following calculation, similar to the one in Lemma 7.5. We have

$$\hat{Z}^I\widetilde{\Box}_g\phi = \hat{Z}^I\Box\phi + \hat{Z}^I H^{\alpha\beta}\partial^2_{\alpha\beta}\phi = \Box Z^I\phi + \sum_{J+K=I}\hat{Z}^J H^{\alpha\beta} Z^K\partial^2_{\alpha\beta}\phi$$

If we now use (7.16) we get (7.19) as in the proof of Lemma 7.5. The inequality (7.21) now follows from (7.19), (7.13) and the estimate (7.7). ☐

8 Basic energy identities

We now establish basic energy identities for solutions of the equation

$$\widetilde{\Box}_g\phi = F \tag{8.1}$$

We denote by Σ_t the hyper surfaces t =const, by $C_{t_1}^{t_2}(q)$ the forward light cones with a vertex at $(q,0)$ and truncated at times t_1, t_2. We also denote by $K_{t_1}^{t_2}(q)$ the interior of the light cone $C_{t_1}^{t_2}(q)$and by $B_{t,r}$ the ball of radius r centered at $(t,0)$.

Lemma 8.1. *Let ϕ be a solution of* (8.1)*. Then for any $t_1\le t_2$ and an arbitrary $q\le t_2$*

$$\begin{aligned}\int_{\Sigma_{t_2}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big) &= \int_{\Sigma_{t_1}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big)\\ &\quad - 2\int_{t_1}^{t_2}\int_{\Sigma_\tau}\Big(\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\Big),\end{aligned} \tag{8.2}$$

and

$$\begin{aligned}\int_{B_{t_1-q}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big) + \int_{C_{t_1}^{t_2}(q)}|\bar\partial\phi|^2 &= \int_{B_{t_2-q}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big)\\ &\quad + 2\int_{K_{t_1}^{t_2}(q)}\Big(\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\Big)\\ &\quad + 2\int_{C_{t_1}^{t_2}(q)}\big(2(g^{\alpha\beta}-m^{\alpha\beta})L_\alpha\partial_\beta\phi\partial_t\phi + (g^{\alpha\beta}-m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi\big)\end{aligned} \tag{8.3}$$

Proof. We multiply the equation (8.1) by $\partial_t\phi$ and integrate over the space-time slab between the hyper surfaces Σ_{t_1} and Σ_{t_2}. We have

$$\begin{aligned}
-\int_{t_1}^{t_2}\int_{\Sigma_\tau} g^{\alpha\beta}\partial^2_{\alpha\beta}\phi\partial_t\phi &= \int_{t_1}^{t_2}\int_{\Sigma_\tau}\big(g^{\alpha\beta}\partial_\beta\phi\partial_t\partial_\alpha\phi + \partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi\big)\\
&\quad - \int_{\Sigma_{t_2}} g^{0\beta}\partial_\beta\phi\partial_t\phi + \int_{\Sigma_{t_1}} g^{0\beta}\partial_\beta\phi\partial_t\phi\\
&= \frac12\int_{\Sigma_{t_2}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big) - \frac12\int_{\Sigma_{t_1}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big)\\
&\quad + \int_{t_1}^{t_2}\int_{\Sigma_\tau}\big(\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac12\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\big)
\end{aligned}$$

and the desired identity (8.2) follows. Similarly, integrating over the region $K_{t_1}^{t_2}(q)$ we obtain

$$\begin{aligned}
\int_{B_{t_1-q}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big) &- \int_{C_{t_1}^{t_2}(q)}\big(2g^{\alpha\beta}L_\alpha\partial_\beta\phi\partial_t\phi + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\big)\\
&= \int_{B_{t_2-q}}\big(-g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big)\\
&\quad + 2\int_{K_{t_1}^{t_2}(q)}\Big(\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac12\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\Big)
\end{aligned}$$

Subtracting the Minkowski part from the metric g in the $C_{t_1}^{t_2}(q)$ integral leads to the identity (8.3). $\square$

Corollary 8.2. *Let ϕ be a solution of the equation* (8.1) *with a metric g satisfying the condition that*

$$|H| \le \frac14, \qquad H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}. \tag{8.4}$$

Then for any $0 < \gamma \le 1$

$$\begin{aligned}
\int_{\Sigma_{t_2}}\big(|\partial_t\phi|^2 + |\nabla_x\phi|^2\big) + \int_{t_1}^{t_2}\int_{\Sigma_\tau}\frac{\gamma\,|\bar\partial\phi|^2}{(1+|q|)^{1+2\gamma}} &\le 4\int_{\Sigma_{t_1}}\big(|\partial_t\phi|^2 + |\nabla_x\phi|^2\big)\\
&\quad + 8\int_{t_1}^{t_2}\int_{\Sigma_\tau}\big|\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac12\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\big|\\
&\quad + 2\int_{t_1}^{t_2}\int_{\Sigma_\tau}\frac{\gamma}{(1+|q|)^{1+2\gamma}}\big|(g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi + 2(g^{\underline{L}\beta} - m^{\underline{L}\beta})\partial_\beta\phi\partial_t\phi\big|
\end{aligned} \tag{8.5}$$

Proof. First we note that (8.4) implies that

$$\frac34\big(|\partial_t\phi|^2 + |\nabla_x\phi|^2\big) \le -g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\,\partial_j\phi \le \frac54\big(|\partial_t\phi|^2 + |\nabla_x\phi|^2\big) \tag{8.6}$$

The inequalities (8.3) and (8.2)imply that

$$\int_{C_{t_1}^{t_2}(q)} |\bar\partial\phi|^2 \le \int_{\Sigma_{t_2}} \big(- g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big) \tag{8.7}$$

$$+ 2\int_{K_{t_1}^{t_2}(q)} \partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi$$

$$+ 2\int_{C_{t_1}^{t_2}(q)} 2(g^{\alpha\beta} - m^{\alpha\beta})L_\alpha\partial_\beta\phi\partial_t\phi + (g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi \tag{8.8}$$

$$\le \int_{\Sigma_{t_1}} \big(- g^{00}|\partial_t\phi|^2 + g^{ij}\partial_i\phi\partial_j\phi\big) \tag{8.9}$$

$$+ 2\int_{t_1}^{t_2}\int_{\Sigma_t} \Big|\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\Big|$$

$$+ 2\int_{C_{t_1}^{t_2}(q)} \Big|2(g^{\alpha\beta} - m^{\alpha\beta})L_\alpha\partial_\beta\phi\partial_t\phi + (g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi\Big|$$

We multiply the above inequality by an integrable factor $\gamma(1+|q|)^{-1-2\gamma}$ and integrate with respect to q in the interval $(-\infty, t_2]$ to obtain:

$$\int_{t_1}^{t_2}\int_{\Sigma_t} \frac{\gamma|\bar\partial\phi|^2}{(1+|q|)^{1+2\gamma}} \le \frac{5}{4}\int_{\Sigma_{t_1}} \big(|\partial_t\phi|^2 + |\nabla_x\phi|^2\big)$$

$$+ 2\int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\big| \tag{8.10}$$

$$+ 2\int_{t_1}^{t_2}\int_{\Sigma_\tau} \frac{\gamma}{(1+|q|)^{1+2\gamma}}\big|(g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi + 2(g^{\underline{L}\beta} - m^{\underline{L}\beta})\partial_\beta\phi\partial_t\phi\big|$$

where we also used (8.6). On the other hand using (8.6) and (8.2) yields

$$\int_{\Sigma_{t_1}} \big(|\partial_t\phi|^2 + |\nabla_x\phi|^2\big) \le \frac{5}{3}\int_{\Sigma_{t_1}} \big(|\partial_t\phi|^2 + |\nabla_x\phi|^2\big) \tag{8.11}$$

$$+ \frac{8}{3}\int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\big| \tag{8.12}$$

and the corollary follows. □

9 Poincaré and Klainerman-Sobolev inequalities

We now state the following useful version of the Poincaré inequality.

Lemma 9.1. *Let f be a smooth function. Then for any $\gamma > -1/2, \gamma \ne 1/2$ and any positive t*

$$\int\limits_{R^3} \frac{|f(x)|^2\, dx}{(1+|t-r|)^{2+2\gamma}} \le C \int\limits_{S_{(t+1)}} |f|^2\, dS + C\int\limits_{R^3} \frac{|\partial_r f(x)|^2\, dx}{(1+|t-r|)^{2\gamma}} \tag{9.1}$$

provided that the left hand side is bounded. Here $S_{(t+1)}$ is the sphere of radius $t+1$ and $r = |x|$.

Proof. Using polar coordinates $x = r\omega$ we write

$$|f(r,\omega)|^2 - |f(t+1,\omega)|^2 = -2\int_r^{t+1} \partial_r f(\rho,\omega)\cdot f(\rho,\omega)\, d\rho$$

Hence

$$|f(r,\omega)|^2 r^2 \lesssim |f(t+1,\omega)|^2 (t+1)^2 + 2\int_r^{t+1} |\partial_r f(\rho,\omega)|\, |f(\rho,\omega)|\, \rho^2\, d\rho, \qquad \text{if} \quad r \le t+1.$$

Therefore multiplying by $(1+|t-r|)^{-2-2\gamma}$ and integrating with respect to r from 0 to $t+1$:

$$\begin{aligned}
\int_0^{t+1} \frac{|f(r,\omega)|^2\, r^2\, dr}{(1+|t-r|)^{2+2\gamma}} &\lesssim \int_0^{t+1} \frac{|f(t+1,\omega)|^2\, (t+1)^2\, dr}{(1+|t-r|)^{2+2\gamma}} + \int_0^{t+1}\int_r^{t+1} \frac{|\partial_r f(\rho,\omega)||f(\rho,\omega)|}{(1+|t-r|)^{2+2\gamma}}\, \rho^2\, d\rho\, dr \\
&\lesssim |f(t+1,\omega)|^2\, (t+1)^2 + \int_0^{t+1}\int_0^{\rho} \frac{|\partial_r f(\rho,\omega)||f(\rho,\omega)|}{(1+|t-r|)^{2+2\gamma}}\, dr\, \rho^2\, d\rho \\
&\lesssim |f(t+1,\omega)|^2\, (t+1)^2 + \int_0^{t+1} \frac{|\partial_r f(\rho,\omega)||f(\rho,\omega)|}{(1+|t-\rho|)^{1+2\gamma}}\, \rho^2\, d\rho \\
&\lesssim |f(t+1,\omega)|^2 (t+1)^2 + \Big(\int_0^{t+1} \frac{|\partial_r f(\rho,\omega)|^2\, \rho^2\, d\rho}{(1+|t-\rho|)^{2\gamma}}\Big)^{1/2} \Big(\int_0^{t+1} \frac{|f(\rho,\omega)|^2\, \rho^2\, d\rho}{(1+|t-\rho|)^{2+2\gamma}}\Big)^{1/2},
\end{aligned}$$

where we first changed the order of integration and then used Cauchy-Schwarz inequality. It therefore follows that

$$\int_0^{t+1} \frac{|f(r,\omega)|^2\, r^2\, dr}{(1+|t-r|)^{2+2\gamma}} \lesssim |f(t+1,\omega)|^2\, (t+1)^2 + \int_0^{t+1} \frac{|\partial_r f(\rho,\omega)|^2\, \rho^2\, d\rho}{(1+|t-\rho|)^{2\gamma}}$$

and if we also integrate over the angular variables we get

$$\int\limits_{|x|\le(t+1)} \frac{|f(x)|^2\, dx}{(1+|t-r|)^{2+2\gamma}} \lesssim \int\limits_{S_{(t+1)}} |f|^2 dS + \int\limits_{|x|\le(t+1)} \frac{|\partial_r f(x)|^2\, dx}{(1+|t-r|)^{2\gamma}}$$

On the other hand, if we instead integrate from $t+1$ to $2(t+1)$ we similarly obtain

$$\begin{aligned}
\int_{t+1}^{2(t+1)} \frac{|f(r,\omega)|^2\, r^2\, dr}{(1+|t-r|)^{2+2\gamma}} &\lesssim \int_{t+1}^{2(t+1)} \frac{|f(t+1,\omega)|^2\, (t+1)^2\, dr}{(1+|t-r|)^{2+2\gamma}} + \int_{t+1}^{2(t+1)}\int_{t+1}^{r} \frac{|\partial_r f(\rho,\omega)||f(\rho,\omega)|}{(1+|t-r|)^{2+2\gamma}}\, \rho^2\, d\rho\, dr \\
&\lesssim |f(t+1,\omega)|^2\, (t+1)^2 + \int_{t+1}^{2(t+1)}\int_{\rho}^{2(t+1)} \frac{|\partial_r f(\rho,\omega)||f(\rho,\omega)|}{(1+|t-r|)^{2+2\gamma}}\, dr\, \rho^2\, d\rho \\
&\lesssim |f(t+1,\omega)|^2\, (t+1)^2 + \int_{t+1}^{2(t+1)} \frac{|\partial_r f(\rho,\omega)||f(\rho,\omega)|}{(1+|t-\rho|)^{1+2\gamma}}\, \rho^2\, d\rho
\end{aligned}$$

and as before it follows that

$$\int\limits_{(t+1)\le|x|\le 2(t+1)} \frac{|f(x)|^2\, dx}{(1+|t-r|)^{2+2\gamma}} \lesssim \int\limits_{S_{(t+1)}} |f|^2\, dS + \int\limits_{(t+1)\le|x|\le 2(t+1)} \frac{|\partial_r f(x)|^2\, dx}{(1+|t-r|)^{2\gamma}}$$

Finally, in the region $r \geq 2(t+1)$ the estimate (9.1) would follow from the Hardy type inequality:

$$\int\limits_{|x|\geq(t+1)} \frac{|f(x)|^2\,dx}{|x|^{2+2\gamma}} \leq \int\limits_{|x|\geq(t+1)} \frac{|\partial_r f(x)|^2\,dx}{|x|^{2\gamma}} + (t+1)^{-1-2\gamma} \int\limits_{S_{(t+1)}} |f|^2\,dS, \tag{9.2}$$

that hold provided the left hand side is bounded. One can for the proof assume that f ha compact support since we can choose a sequence of compactly supported functions converging to a given function f in the norm defined by the right hand side as long as the norm in the left of f is bounded. (9.2) for compactly supported smooth functions can be easily seen from integrating the identity

$$\partial_r \left(\frac{r^2 f^2}{r^{1+2\gamma}} \right) = \frac{2r^2}{r^{1+2\gamma}} f \cdot \partial_r f + (1-2\gamma) \frac{r^2}{r^{2+2\gamma}} f^2, \qquad \gamma \neq -1/2$$

from $r = t+1$ to $r = \infty$ and using Cauchy-Schwarz as above. □

We now state the global Sobolev inequality, which is due to S. Klainerman [K1].

Proposition 9.2. *The following inequality holds for an arbitrary smooth function ϕ.*

$$|\phi(t,x)|(1+t+|t-r|)(1+|t-r|)^{1/2} \leq C \sum_{|I|\leq 3} \|Z^I \phi(t,\cdot)\|_{L^2}.$$

10 Decay estimates for the wave equation on a curved space time

In this section we will derive some basic estimates for the scalar wave equation on a curved background. The results will require some weak assumptions on the metric g, which will be easily verified in the case of a metric satisfying the reduced Einstein equations.

We consider the reduced scalar wave equation:

$$\widetilde{\Box}_g \phi = F. \tag{10.1}$$

The following result is a generalization of the lemma in [L1] to the variable coefficient case:

Lemma 10.1. *Suppose that ϕ satisfies the reduced scalar wave equation* (10.1) *on a curved background with a metric g. Suppose that $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ satisfies*

$$|H| \leq \frac{1}{4}, \qquad \textit{and} \qquad |H|_{L\mathcal{T}} \leq \frac{1}{4} \frac{|q|+1}{1+t+|x|}. \tag{10.2}$$

when $t/2 \leq |x| \leq 2t$ and

$$\int_0^\infty \| H(t,\cdot)\|_{L^\infty(D_t)} \frac{dt}{1+t} \leq \frac{1}{4}, \qquad \textit{where} \qquad D_t = \{x \in \mathbf{R}^3;\ t/2 \leq |x| \leq 2t\}. \tag{10.3}$$

Then for any $t \geq 0$ and $x \in \mathbb{R}^3$,

$$\begin{aligned}(1+t+|x|)\,|\partial\phi(t,x)| &\leq C \sup_{0\leq\tau\leq t} \sum_{|I|\leq 1} \|Z^I \phi(\tau,\cdot)\|_{L^\infty} \\ &+ C \int_0^t \Big((1+\tau)\|F(\tau,\cdot)\|_{L^\infty(D_\tau)} + \sum_{|I|\leq 2} (1+\tau)^{-1} \|Z^I \phi(\tau,\cdot)\|_{L^\infty(D_\tau)} \Big)\, d\tau\end{aligned} \tag{10.4}$$

Proof. Since by Lemma 7.2

$$(1+|t-r|)|\partial\phi| + (1+t+r)|\bar\partial\phi| \le C\sum_{|I|=1}|Z^I\phi|, \qquad r=|x|, \tag{10.5}$$

the inequality (10.4) holds when $r<t/2+1/2$ or $r>2t-1$. Furthermore, since

$$(1+r)|\partial_q\phi| \le C|\partial_q(r\phi)| + C|\phi|, \qquad r\ge 1 \tag{10.6}$$

it follows that

$$(1+t+r)|\partial\phi| \le C\sum_{|I|\le 1}|Z^I\phi| + C|\partial_q(r\phi)| \tag{10.7}$$

Hence it suffices to prove that $|\partial_q(r\phi)|$ is bounded by the right hand side of (10.4) when $t/2+1/2 < r < 2t-1$. By Lemma 7.3

$$\big|(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q)\partial_q(r\phi)\big| \lesssim \Big(1+\frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|}+|H|\Big)r^{-1}\sum_{|I|\le 2}|Z^I\phi| + |H|\,r^{-1}\,|\partial_q(r\phi)| + r|F| \tag{10.8}$$

and using the decay assumptions (10.2) and (10.7) we get

$$\big|(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q)\partial_q(r\phi)\big| \lesssim \frac{|H|}{1+t}|\partial_q(r\phi)| + \sum_{|I|\le 2}\frac{Z^I\phi|}{1+t} + C(t+1)|F|, \qquad \text{when} \quad t/2+1/2\le r\le 2t-1 \tag{10.9}$$

Along an integral curve $(t,x(t))$ of the vector field $\partial_s + H^{\underline{LL}}(2g^{L\underline{L}})^{-1}\partial_q$, contained in the region $t/2+1/2\le |x|\le 2t-1$, we have the following equation for $\psi=\partial_q(r\phi)$:

$$\Big|\frac{d}{dt}\psi\Big| \le \hat h|\psi| + f \tag{10.10}$$

where $\hat h = C|H|/(1+t)$ and $f = Ct|F| + C\sum_{|I|\le 2}|Z^I\phi|/(1+t)$. Hence multiplying (10.10) with the integrating factor $e^{-\hat H}$, where $\hat H = \int \hat h(s)\,ds$ we get

$$\Big|\frac{d}{dt}\Big(\psi e^{-\hat H}\Big)\Big| \le f e^{-\hat H}. \tag{10.11}$$

If we integrate backwards along an integral curve from any point (t,x) in the set $t/2+1/2\le|x|\le 2t-1$ until the first time the curve intersects the boundary of the set at (τ,y), $|y|=\tau/2+1/2$ or $|y|=2\,\tau-1$, we obtain

$$|\psi(t,x)| \le \exp\Big(\int_\tau^t \|\hat h(\sigma,\cdot)\|_{L^\infty}\,d\sigma\Big)|\psi(\tau,y)| + \int_\tau^t \exp\Big(\int_{\tau'}^t \|\hat h(\sigma,\cdot)\|_{L^\infty}\,d\sigma\Big)\|f(\tau',\cdot)\|_{L^\infty}\,d\tau',$$

where the L^∞ norms are taken only over the set $t/1+1/2\le|x|\le 2t-1$. (Note that any integral curve has to intersect either of the two boundaries $r=t/2+1/2$ or $r=2t-1$ since the slope of the curve $x(t)$ has to be close to 1 when H_{LL} is small.) The lemma now follows from taking the supremum over x in the set $t/2+1/2\le|x|\le 2t-1$, using that on the cones $|y|=\tau/2+1/2$ or $|y|=2\tau-1$ we have that $|\psi|\le Cr|\partial_q\phi| + C|\phi| \le C\sum_{|I|\le 1}|Z^I\phi|$, by (10.5), and using that by (10.3) $\int_0^t \|\hat h(\sigma,\cdot)\|_{L^\infty}\,d\sigma \le \frac14$. $\square$

For second order derivatives we have an estimate which gives a slightly worse decay:

Lemma 10.2. *Let ϕ be a solution of the reduced scalar wave equation on a curved background with a metric g. Assume that $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ satisfies*

$$\sum_{|I|\le 1} |Z^I H| \le \frac{\tilde{\varepsilon}}{4}, \qquad \textit{and} \qquad \sum_{|I|\le 1} |Z^I H|_{LL} + |H|_{L\mathcal{T}} \le \frac{\tilde{\varepsilon}}{4}\,\frac{|q|+1}{1+t+|x|}. \tag{10.12}$$

when $t/2 \le |x| \le 2t$ for some $\tilde{\varepsilon} \le 1$. Then, for $t \ge 0$, $x \in \mathbb{R}^3$, we have

$$\begin{aligned}(1+t+|x|)\sum_{|I|\le 1} |\partial Z^I \phi(t,x)| &\le C \sup_{0\le\tau\le t} \Big(\frac{1+t}{1+\tau}\Big)^{C\tilde{\varepsilon}} \sum_{|I|\le 2} \|Z^I\phi(\tau,\cdot)\|_{L^\infty}\Big) \\ &+ C\int_0^t \Big(\frac{1+t}{1+\tau}\Big)^{C\tilde{\varepsilon}} \Big(\sum_{|I|\le 1}(1+\tau)\|Z^I F(\tau,\cdot)\|_{L^\infty(D_\tau)} + \sum_{|I|\le 3}(1+\tau)^{-1}\|Z^I\phi(\tau,\cdot)\|_{L^\infty(D_\tau)}\Big)\, d\tau,\end{aligned} \tag{10.13}$$

where $D_t = \{x \in \mathbf{R}^3;\, t/2 \le |x| \le 2t\}$.

Proof. First when $r < t/2$ or $r > t/2$ the lemma trivially follows from (10.5) with ϕ replaced by $Z\phi$ so it only remains to prove the lemma when $t/2 \le r \le 2t$. We have

$$\widetilde{\square}_g Z\phi = F_Z = \hat{Z}F + \big(\widetilde{\square}_g Z\phi - \hat{Z}\widetilde{\square}_g\phi\big), \tag{10.14}$$

where by (7.18) the additional commutator term can be estimated by

$$\big|\widetilde{\square}_g Z\phi - \hat{Z}\widetilde{\square}_g\phi\big| \lesssim \Big(\frac{|ZH|+|H|}{1+t+|q|} + \frac{|ZH|_{LL}+|H|_{L\mathcal{T}}}{1+|q|}\Big)\sum_{|I|\le 1}|\partial Z^I\phi| \lesssim \frac{\tilde{\varepsilon}}{1+t+q}\sum_{|I|\le 1}|\partial Z^I\phi|, \tag{10.15}$$

where we used the decay assumption (10.12). Furthermore with the help of (10.7), applied to $Z^I\phi$ in place of ϕ, we obtain

$$\big|\widetilde{\square}_g Z\phi - \hat{Z}\widetilde{\square}_g\phi\big| \lesssim \frac{\tilde{\varepsilon}}{(1+t+|q|)^2}\Big(\sum_{|I|\le 1}\big|\partial_q(rZ^I\phi)\big| + \sum_{|I|\le 2}|Z^I\phi|\Big), \tag{10.16}$$

Hence by (10.9) applied to (10.14) in place of (10.1) we get

$$|(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q)\partial_q(rZ\phi)| \lesssim \sum_{|I|\le 3}\frac{|Z^I\phi|}{1+t} + \frac{\tilde{\varepsilon}}{1+t}\sum_{|I|\le 1}|\partial_q(rZ^I\phi)| + t(|ZF|+|F|) \tag{10.17}$$

when $t/2 + 1/2 \le r \le 2t-1$. Therefore

$$\Big|(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q)\sum_{|I|\le 1}|\partial_q(rZ^I\phi)|\Big| \le \frac{C\tilde{\varepsilon}}{1+t}\sum_{|I|\le 1}|\partial_q(rZ^I\phi)| + C\sum_{|I|\le 3}\frac{|Z^I\phi|}{1+t} + Ct(|ZF|+|F|) \tag{10.18}$$

The desired result follows multiplying (10.17) by the factor $(1+t)^{-C\tilde{\varepsilon}}$ and integrating as in the proof of the previous lemma. Along an integral curve we have the equation

$$\Big|\frac{d}{dt}\Big(\psi(1+t)^{-C\tilde{\varepsilon}}\Big)\Big| \le (1+t)^{-C\tilde{\varepsilon}} f, \tag{10.19}$$

where

$$\psi = \sum_{|I|\le 1} |\partial_q(Z^I\phi)|, \quad f = C(1+t)(|ZF|+|F|) + C\sum_{|I|\le 3}\frac{|Z^I\phi|}{1+t} \tag{10.20}$$

The lemma now follows as in the proof of Lemma 10.1. $\square$

We observe that similar estimates hold for a system

$$\widetilde{\square}_g \phi_{\mu\nu} = F_{\mu\nu} \tag{10.21}$$

In particular, in our case, certain components of $F_{\mu\nu}$ expressed in the null-frame will decay better than others and for these components we will also get better estimates for $\phi_{\mu\nu}$. Since the vector fields L and $\underline{L}$ commute with contractions of any of the vector fields $\{L, \underline{L}, S_1, S_2\}$ proofs of the preceding lemmas imply the following result:

Corollary 10.3. *Let $\phi_{\mu\nu}$ be a solution of reduced wave equation system* (10.21) *on a curved background with a metric g. Assume that $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ satisfies*

$$\sum_{|I|\le 1} |Z^I H| \le \frac{\tilde\varepsilon}{4}, \qquad \textit{and} \qquad \sum_{|I|\le 1} |Z^I H|_{LL} + |H|_{L\mathcal{T}} \le \frac{\tilde\varepsilon}{4}\frac{|q|+1}{1+t+|x|}. \tag{10.22}$$

when $t/2\le |x|\le 2t$, for some $\tilde\varepsilon\le 1$ and

$$\int_0^\infty \| H(t,\cdot)\|_{L^\infty(D_t)}\frac{dt}{1+t} \le \frac{\tilde\varepsilon}{4}. \tag{10.23}$$

where $D_t = \{x\in\mathbf{R}^3;\, t/2\le |x|\le 2t\}$. Then for any $U,V\in\{L,\underline{L},S_1,S_2\}$ and any $t\ge 0$, $x\in\mathbb{R}^3$:

(10.24)

$$\begin{aligned}(1+t+|x|)\ |\partial\phi(t,x)|_{UV} &\le C\sup_{0\le\tau\le t}\sum_{|I|\le 1}\|Z^I\phi(\tau,\cdot)\|_{L^\infty}\\ &+ C\int_0^t \Big((1+\tau)\||F|_{UV}(\tau,\cdot)\|_{L^\infty(D_\tau)} + \sum_{|I|\le 2}(1+\tau)^{-1}\|Z^I\phi(\tau,\cdot)\|_{L^\infty(D_\tau)}\Big)\,d\tau,\end{aligned}$$

(10.25)

$$\begin{aligned}(1+t+|x|)\sum_{|I|\le 1}|\partial Z^I\phi|(t,x)| &\le C\sup_{0\le\tau\le t}\Big(\frac{1+t}{1+\tau}\Big)^{C\tilde\varepsilon}\sum_{|I|\le 2}\|Z^I\phi(\tau,\cdot)\|_{L^\infty}\\ &+ C\int_0^t\Big(\frac{1+t}{1+\tau}\Big)^{C\tilde\varepsilon}\Big(\sum_{|I|\le 1}\|r(\cdot)|Z^IF|(\tau,\cdot)\|_{L^\infty(D_\tau)} + \sum_{|I|\le 3}(1+\tau)^{-1}\|Z^I\phi(\tau,\cdot)\|_{L^\infty(D_\tau)}\Big)\,d\tau.\end{aligned}$$

Proof. By Lemma 7.3 for each component we have the estimate

$$\Big|\Big(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q - \frac{\overline{\operatorname{tr}}\,H\ + H_{L\underline{L}}}{2g^{L\underline{L}}\ r}\Big)\partial_q(r\phi_{\mu\nu}) + \frac{rF_{\mu\nu}}{2g^{L\underline{L}}}\Big| \lesssim \Big(1+\frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|}+|H|\Big)r^{-1}\sum_{|I|\le 2}|Z^I\phi_{\mu\nu}| \tag{10.26}$$

and since ∂_s and ∂_q commute with contractions with the frame vectors $L, \underline{L}$ we get

$$\Big|\Big(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q - \frac{\overline{\operatorname{tr}}\, H\ + H_{L\underline{L}}}{2g^{L\underline{L}}\ r}\Big)\partial_q(r\phi_{UV}) + \frac{rF_{UV}}{2g^{L\underline{L}}}\Big| \lesssim \Big(1 + \frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|} + |H|\Big)r^{-1}\sum_{|I|\le 2}|Z^I\phi| \tag{10.27}$$

As before it also follows that

$$(1+t+|r)|\partial\phi|_{UV} \lesssim \sum_{|I|\le 1}|Z^I\phi| + |\partial_q(r\phi)|_{UV} \tag{10.28}$$

The lemma now follows as before. □

11 Energy estimates for the wave equation on a curved space time

In this section we derive the energy estimate for a solution ϕ of the inhomogeneous wave equation

$$\widetilde{\Box}_g\phi = F \tag{11.1}$$

under the following assumptions on the metric $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$:

$$\begin{aligned}(1+|q|)^{-1}|H|_{LL} + |\partial H|_{LL} + |\overline{\partial} H| &\le C\varepsilon(1+t)^{-1},\\ (1+|q|)^{-1}\,|H| + |\partial H| &\le C\varepsilon(1+t)^{-\frac12}(1+|q|)^{-\frac12-\gamma}\end{aligned} \tag{11.2}$$

Proposition 11.1. *Let ϕ be a solution of the wave equation* (11.1) *with the metric g verifying the assumptions* (11.2)*. Then for any $0<\gamma\le 1/2$, there is an ε_0 such that for $\varepsilon<\varepsilon_0$*

$$\int_{\Sigma_t}|\partial\phi|^2 + \int_0^t\int_{\Sigma_\tau}\frac{\gamma\,|\bar\partial\phi|^2}{(1+|q|)^{1+2\gamma}} \le 8\int_{\Sigma_0}|\partial\phi|^2 + C\varepsilon\int_0^t\int_{\Sigma_t}\frac{|\partial\phi|^2}{1+t} + 16\int_0^t\int_{\Sigma_t}|F||\partial_t\phi| \tag{11.3}$$

Remark 11.2. Observe that by the Gronwall inequality the energy estimate of the above proposition implies t^ε growth of the energy. For similar estimates, proved under different assumptions, see also [S1],[A2],[A3].

Proof. The proof of the proposition relies on the energy estimate obtained in Corollary 8.2

$$\begin{aligned}\int_{\Sigma_t}\big(|\partial_t\phi|^2+|\nabla\phi|^2\big) + \int_0^t\int_{\Sigma_\tau}\frac{\gamma\,|\bar\partial\phi|^2}{(1+|q|)^{1+2\gamma}} &\le 4\int_{\Sigma_0}\big(|\partial_t\phi|^2+|\nabla\phi|^2\big)\\ &+8\int_0^t\int_{\Sigma_\tau}\big|\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac12\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + F\partial_t\phi\big|\\ &+2\int_0^t\int_{\Sigma_\tau}\frac{\gamma}{(1+|q|)^{1+2\gamma}}\big|(g^{\alpha\beta}-m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi + 2(g^{\underline{L}\beta}-m^{\underline{L}\beta})\partial_\beta\phi\partial_t\phi\big|\end{aligned}$$

We start by decomposing the terms on the right hand side with respect to the null frame.

$$|\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi| \le \big(|H|\,|\partial H| + |(\partial H)_{LL}| + |\bar\partial H|\big)\,|\partial\phi|^2 + |\partial H|\,|\bar\partial\phi|\,|\partial\phi|$$

Similarly,

$$|\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi| \le \big(|g-m|\,|\partial g| + |(\partial g)_{LL}| + |\bar\partial g|\big)\,|\partial\phi|^2 + |\partial g|\,|\bar\partial\phi|\,|\partial\phi|$$

Therefore, using the assumptions (11.2) on the metric g, we obtain that

$$|\partial_\alpha g^{\alpha\beta}\partial_\beta\phi\partial_t\phi - \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi| \lesssim \frac{\varepsilon}{1+t}|\partial\phi|^2 + \frac{\varepsilon}{(1+|q|)^{1+2\gamma}}|\bar\partial\phi|^2 \tag{11.4}$$

Decomposing the remaining terms we infer that

$$|(g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi| \le |H_{LL}|\,|\partial\phi|^2 + |H||\bar\partial\phi|\,|\partial\phi|$$

Similarly,

$$|(g^{\alpha\beta} - m^{\alpha\beta})L_\alpha\partial_\beta\phi\partial_t\phi| \le |H_{LL}|\,|\partial\phi|^2 + |H|\,|\bar\partial\phi|\,|\partial\phi|$$

Once again, using the assumptions (11.2), we have

$$|2(g^{\alpha\beta} - m^{\alpha\beta})L_\alpha\partial_\beta\phi\partial_t\phi + (g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi| \lesssim \varepsilon\frac{1+|q|}{1+t}|\partial\phi|^2 + \frac{\varepsilon}{(1+|q|)^{2\gamma}}|\bar\partial\phi|^2 \tag{11.5}$$

Thus

$$\int_{\Sigma_t}|\partial\phi|^2 + \int_0^t\int_{\Sigma_\tau}\frac{\gamma\,|\bar\partial\phi|^2}{(1+|q|)^{1+2\gamma}} \le 4\int_{\Sigma_0}|\partial\phi|^2 + C\varepsilon\int_0^t\int_{\Sigma_\tau}\Big(\frac{|\partial\phi|^2}{1+t} + \frac{|\bar\partial\phi|^2}{(1+|q|)^{1+2\gamma}}\Big) + 8\int_0^t\int_{\Sigma_\tau}|F|\,|\partial_t\phi|$$

and the desired estimate follows if we take ε so small that $C\varepsilon < \gamma/2$. □

12 Estimates from the wave coordinate condition

In previous sections we have shown that one only needs to control certain components of the metric in order to establish decay estimates for solutions of the reduced wave equation. In this section we will see that the wave coordinate condition allows one to estimate precisely those components in terms of tangential derivatives or higher order terms with better decay better. Recall that the wave coordinate condition can be written in the form

$$\partial_\mu\Big(g^{\mu\nu}\sqrt{|\det g|}\Big) = 0 \tag{12.1}$$

We have the following decomposition

$$g^{\mu\nu}\sqrt{|\det g|} = \big(m^{\mu\nu} + H^{\mu\nu}\big)\big(1 - \frac{1}{2}\mathrm{tr}H + O(H^2)\big)$$

where $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$, $h_{\alpha\beta} = g_{\alpha\beta} - m_{\alpha\beta}$. Recall also that $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ and $H^{\alpha\beta} = -m^{\mu\alpha}m^{\nu\beta}h_{\mu\nu} + O(h^2)$. Therefore we obtain the following expression for the wave coordinate condition:

$$\partial_\mu\Big(H^{\mu\nu} - \frac{1}{2}m^{\mu\nu}\,\mathrm{tr}\ H + O^{\mu\nu}(H^2)\Big) = 0 \tag{12.2}$$

Using that we can express the divergence in terms of the null frame

$$\partial_\mu F^\mu = L_\mu\partial_q F^\mu - \underline{L}_\mu\partial_s F^\mu + A_\mu\partial_A F^\mu \tag{12.3}$$

we obtain:

Lemma 12.1. *Assume that $|H| \le 1/4$. Then*

$$|\partial H|_{L\mathcal{T}} \lesssim |\overline{\partial} H| + |H|\,|\partial H|, \qquad |\partial\, \overline{\mathrm{tr}}\, H| \lesssim |\bar\partial H| + |H|\,|\partial H| \tag{12.4}$$

Proof. It follows from (12.2) and (12.3) that

$$\big|L_\mu \partial\big(H^{\mu\nu} - \frac{1}{2} m^{\mu\nu}\,\mathrm{tr}\; H\big)\big| \le |\bar\partial H| + |H||\partial H| \tag{12.5}$$

Contracting with T_ν and using that $m_{TL} = 0$ gives the first inequality and contracting with $\underline{L}_\mu$ and using that $m_{\underline{L}L} = -2$ gives the second since

$$H_{L\underline{L}} + \mathrm{tr}\; H = \overline{\mathrm{tr}}\, H \tag{12.6}$$

□

We now compute the commutators of the wave coordinate condition with the vector fields Z.

Lemma 12.2. *Let Z be one of the Minkowski Killing or conformally Killing vector fields and let tensor H satisfy the wave coordinate condition. Then the estimate*

$$\big|\partial H^J\big|_{L\mathcal{T}} \lesssim \sum_{|J|\le|I|} |\bar\partial Z^J H| + \sum_{I_1+\ldots+I_k=I,\, k\ge 2} |Z^{I_k} H| \cdots |Z^{I_2} H|\,|\partial Z^{I_1} H|$$

holds true for the expression

$$H^J_{\mu\nu} = Z^J \widetilde{H}_{\mu\nu} + \sum_{|J|<|I|} c^{I\ \ \gamma}_{J\,\mu} Z^J \widetilde{H}_{\gamma\nu}, \qquad \textit{where} \qquad \widetilde{H}_{\mu\nu} = H_{\mu\nu} - \frac{1}{2} m_{\mu\nu}\, trH \tag{12.7}$$

with some constant tensors $c^{I\,\gamma\mu}_J$ such that $c^{I\ \ \underline{L}}_{J L} = 0$ if $|J| = |I| - 1$.

Proof. The wave coordinate condition (12.1) can be written in the form

$$\partial_\mu\big(\widetilde{G}^{\mu\nu}\big) = 0, \qquad \text{where} \quad \widetilde{G}^{\mu\nu} = (m^{\mu\nu} + H^{\mu\nu})\sqrt{|\det g|}.$$

Let Z be one of the Minkowski Killing or conformally Killing vector fields. Then for any vector field F we have that

$$Z^I \partial_\alpha F^\alpha = \partial_\alpha\Big(Z^I F^\alpha + \sum_{|J|<|I|} c^{I\ \alpha}_{J\ \gamma} Z^J F^\gamma\Big) = \partial_\alpha\Big(\sum_{|J|\le|I|} c^{I\ \alpha}_{J\ \gamma} Z^J F^\gamma\Big),$$

where $c^{\ \alpha}_{J\ \gamma}$ are constants such that

$$c^{I\ \alpha}_{J\ \gamma} = \delta^\alpha_{\ \gamma}, \quad \text{if} \quad |J| = |I| \qquad \text{and} \qquad c^{I\ \underline{L}}_{J\ L} = 0, \quad \text{if} \quad |J| = |I| - 1$$

The last identity is a consequence of the relation between $c^{I\ \gamma}_{J\ \alpha}$ and the commutator constants $c_{\alpha\beta} = [\partial_\alpha, Z]_\beta$ for which we have established that $c_{L\underline{L}} = 0$. It therefore follows that

$$\partial_\mu\big(\sum_{|J|\le|I|} c^{I\ \mu\gamma}_J Z^J \widetilde{G}_{\gamma\nu}\big) = 0.$$

Decomposing relative to the null frame $(L, \underline{L}, S_1, S_2)$ we obtain

$$\partial_q\Big(\sum_{|J|\le|I|} c^I_J{}^{\underline{L}\gamma} Z^J \widetilde{G}_{\gamma\nu}\Big) = \partial_s\Big(\sum_{|J|\le|I|} c^I_J{}^{L\gamma} Z^J \widetilde{G}_{\gamma\nu}\Big) - A_\mu \bar\partial_A\Big(\sum_{|J|\le|I|} c^I_J{}^{\mu\gamma} Z^J \widetilde{G}_{\gamma\nu}\Big).$$

We now contract the above identity with one of the tangential vector fields T^ν, $T\in\{L, S_1, S_2\}$ to obtain

$$\Big| L^\gamma T^\nu \partial_q Z^I \widetilde{G}_{\gamma\nu} + \sum_{|J|<|I|} c^I_J{}^{\underline{L}\gamma} T^\nu \partial_q Z^J \widetilde{G}_{\gamma\nu}\Big| \lesssim \sum_{|J|\le|I|} |\bar\partial Z^I \widetilde{G}|$$

We examine the expression

$$L^\gamma T^\nu Z^J \partial_q \widetilde{G}_{\gamma\nu} = L^\gamma T^\nu \partial_q Z^J\Big((m_{\gamma\nu} + H_{\gamma\nu})\sqrt{|\det g|}\Big) = \sum_{J_1+J_2=J} L^\gamma T^\nu \partial_q \Big((Z^{J_1} H_{\gamma\nu}) Z^{J_2}\sqrt{|\det g|}\Big)$$

since $m_{LT} = L^\gamma T^\nu m_{\gamma\nu} = 0$. The desired estimate now follows from the identity $\sqrt{|\det g|} = 1 + f(H)$, which holds with a smooth function $f(H)$ such that $f(H) = -\mathrm{tr} H/2 + O(H^2)$ $\square$

We now summarize the results of this section in the following

Lemma 12.3. *For a tensor H obeying the wave coordinate condition*

$$|\partial H|_{LT} \lesssim |\bar\partial H| + |H|\,|\partial H|, \tag{12.8}$$

and

$$|\partial Z H|_{LL} \lesssim |\partial H|_{L\mathcal{T}} + \sum_{|I|\le 1} |\overline{\partial} Z^I H| + \sum_{|I|+|J|\le 1} |Z^I H|\,|\partial Z^J H|. \tag{12.9}$$

In general,

$$|\partial Z^I H|_{L\mathcal{T}} \lesssim \sum_{|J|\le|I|} |\overline{\partial} Z^J H| + \sum_{|J|\le|I|-1} |\partial Z^J H| + \sum_{|I_1|+\ldots+|I_m|\le|I|,\, m\ge 2} |Z^{I_m} H| \cdots |Z^{I_2} H||\partial Z^{I_1} H|, \tag{12.10}$$

and

$$|\partial Z^I H|_{LL} \lesssim \sum_{|J|\le|I|} |\overline{\partial} Z^J H| + \sum_{|J|\le|I|-1} |\partial Z^J H|_{L\mathcal{T}} + \sum_{|K|\le|I|-2} |\partial Z^J H| + \sum_{|I_1|+\ldots+|I_m|\le|I|,\, m\ge 2} |Z^{I_m} H| \cdots |Z^{I_2} H||\partial Z^{I_1} H|. \tag{12.11}$$

The same estimates also hold for H replaced by h.

Proof. This follows directly by the previous lemma with the help of the identities $m_{LT} = 0$ and $c^I_J{}^{\underline{L}}_{L} = 0$. $\square$

13 Estimates for the inhomogeneous terms

In this section we will show that the inhomogeneous terms of the reduced Einstein equations can be estimated in terms of tangential derivatives, for which we have better decay estimates, or tangential components which in turn can be expressed, using the wave coordinate condition, in terms of tangential derivatives and lower order terms. Recall that according to Lemma 3.2 the symmetric two tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ verifies the reduced Einstein equations of the form:

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h),$$

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h), \tag{13.1}$$

$$P(\partial_\mu k, \partial_\nu p) = \frac{1}{4}\partial_\mu \mathrm{tr} k\, \partial_\nu \mathrm{tr} p - \frac{1}{2}\partial_\mu k^{\alpha\beta}\partial_\nu p_{\alpha\beta}, \tag{13.2}$$

Here $Q\mu\nu$ are linear combinations of the null-forms and $G_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic form in ∂h with coefficients that are smooth functions of h and vanishing at $h = 0$.

Lemma 13.1. *The quadratic form P satisfies the following pointwise estimate:*

$$|P(\partial p, \partial k)|_{\mathcal{TU}} \lesssim |\bar\partial p\,|\,|\partial k| + |\partial p\,|\,|\bar\partial k|, \tag{13.3}$$

$$|P(\partial p, \partial k)| \lesssim |\partial p\,|_{\mathcal{TU}}|\partial k|_{\mathcal{TU}} + |\partial p\,|_{LL}|\partial k| + |\partial p\,|\,|\partial k|_{LL} \tag{13.4}$$

Proof. The first part of the statement follows trivially from (13.2). To prove (13.4) we use (5.9) applied to $R^\mu\partial_\mu p$ in place of p and $S^\nu\partial_\nu k$ in place of k, for any vector fields T and S, to obtain

$$|T^\mu S^\nu\, P(\partial_\mu p, \partial_\nu k)| \lesssim |T^\mu\partial_\mu p\,|_{\mathcal{TU}}|S^\nu\partial_\nu k|_{\mathcal{TU}} + |T^\mu\partial_\mu p\,|_{LL}|S^\nu\partial_\nu k| + |T^\mu\partial_\mu p\,|\,|S^\nu\partial_\nu k|_{LL} \tag{13.5}$$

which proves the lemma. □

Using the additional estimates on the h_{LL} component, derived in Lemma 12.3 under the assumption that the wave coordinate condition holds, we obtain the following:

Corollary 13.2. *Under the additional assumption that h satisfies the wave coordinate condition* (3.4)*, the quadratic form P obeys the estimate*

$$|P(\partial h, \partial h)|_{\mathcal{TU}} \lesssim |\bar\partial h|\,|\partial h|, \tag{13.6}$$

$$|P(\partial h, \partial h)| \lesssim |\partial h|^2_{\mathcal{TU}} + |\bar\partial h|\,|\partial h| + |h|\,|\partial h|^2 \tag{13.7}$$

Moreover,

$$|Z^I P(\partial h, \partial h)| \lesssim \sum_{|J|+|K|\le|I|}\big(|\partial Z^J h|_{\mathcal{TU}}\,|\partial Z^K h|_{\mathcal{TU}} + |\bar\partial Z^J h|\,|\partial Z^K h|\big) + \sum_{|J|+|K|\le|I|-1}|\partial Z^J h|_{L\mathcal{T}}|\partial Z^K h|$$
$$+ \sum_{|J|+|K|\le|I|-2}|\partial Z^J h|\,|\partial Z^K h| + \sum_{|J_1|+\ldots+|J_m|\le|I|,\, m\ge3}|Z^{J_m} h|\cdots|Z^{J_3}h|\,|\partial Z^{J_2}h||\partial Z^{J_1}h|$$

Proof. The inequality (13.6) follows directly from (13.3). To prove (13.7) we use (13.4) and that by the wave coordinate condition $|\partial h|_{LL} \lesssim |\bar\partial h| + |h|\,|\partial h|$.

We now note that $Z^I P(\partial_\mu h, \partial_\nu h)$ is a sum of terms of the form $P(\partial_\alpha Z^J h, \partial_\beta Z^K h)$ for some α, β and $|J| + |K| \le I$:

$$|Z^I P(\partial h, \partial h)| \le C \sum_{|J|+|K|\le|I|} |P(\partial Z^J h, \partial Z^K h)|$$

It follows from Lemma 5.4 and Lemma 12.3 that

$$\begin{aligned}(13.8)\qquad \sum_{|J|+|K|\le|I|} |P(\partial Z^J h, \partial Z^K h)| &\lesssim \sum_{|J|+|K|\le|I|} |\partial Z^J h|_{\mathcal{TU}} |\partial Z^K h|_{\mathcal{TU}} + |\partial Z^J h|_{LL}\, |\partial Z^K h| \\ &\lesssim \sum_{|J|+|K|\le|I|} |\bar\partial Z^J h|\, |Z^K h| + |\partial Z^J h|_{\mathcal{TU}}\, |\partial Z^K h|_{\mathcal{TU}} \\ + \sum_{|J|+|K|\le|I|} \Big(\sum_{|J'|\le|J|-1} |\partial Z^{J'} h|_{\mathcal{LT}} &+ \sum_{|J''|\le|J|-2} |\partial Z^{J''} h| + \sum_{|J_1|+\ldots+|J_m|\le|J|,\, m\ge2} |Z^{J_m} h| \cdots |Z^{J_2} h|\, |\partial Z^{J_1} h| \Big) |\partial Z^K h|\end{aligned}$$

which proves the lemma. ☐

Proposition 13.3. *Let $F_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h)$ be as in Lemma 3.2 and assume that the wave coordinate condition holds. Then*

$$|F|_{\mathcal{TU}} \lesssim |\overline{\partial} h|\, |\partial h| + |h|\, |\partial h|^2 \tag{13.9}$$

and

$$|F| \lesssim |\partial h|^2_{\mathcal{TU}} + |\bar\partial h|\, |\partial h| + |h|\, |\partial h|^2 \tag{13.10}$$

$$|ZF| \lesssim \big(|\partial h|_{\mathcal{TU}} + |\overline{\partial} h| + |h|\, |\partial h|\big)\big(|\partial Z h| + |\partial h|\big) + \big(|\overline{\partial} Z h| + |Z h|\, |\partial h|\big) |\partial h| \tag{13.11}$$

$$\begin{aligned}(13.12)\qquad |Z^I F| \lesssim &\sum_{|J|+|K|\le|I|} \big(|\partial Z^J h|_{\mathcal{TU}}\, |\partial Z^K h|_{\mathcal{TU}} + |\bar\partial Z^J h|\, |\partial Z^K h|\big) + \sum_{|J|+|K|\le|I|-1} |\partial Z^J h|_{\mathcal{LT}} |\partial Z^K h| \\ &+ \sum_{|J|+|K|\le|I|-2} |\partial Z^J h|\, |\partial Z^K h| + \sum_{|J_1|+\ldots+|J_m|\le|I|,\, m\ge3} |Z^{J_m} h| \cdots |Z^{J_3} h|\, |\partial Z^{J_2} h| |\partial Z^{J_1} h|\end{aligned}$$

Proof. First

$$|Z^I G_{\mu\nu}(h)(\partial h, \partial h)| \le C \sum_{|I_1|+\ldots+|I_k|\le|I|,\, k\ge3} |Z^{I_k} h| \cdots |Z^{I_3} h|\, |\partial Z^{I_2} h|\, |\partial Z^{I_1} h|.$$

Since $ZQ(\partial u, \partial v) = Q(\partial u, \partial Z v) + Q(\partial Z u, \partial v) + a^{ij} Q_{ij}(\partial u, \partial v)$, and $|Q_{\mu\nu}(\partial h, \partial k)| \le |\partial h|\, |\overline{\partial} k| + |\partial k|\, |\overline{\partial} h|$ it follows that

$$|Z^I Q_{\mu\nu}(\partial h, \partial h)| \le C \sum_{|J|+|k|\le|I|} |Q_{\mu\nu}(\partial Z^J h, \partial Z^K h)| \le C \sum_{|J|+|k|\le|I|} |\partial Z^J h|\, |\overline{\partial} Z^K h|$$

☐

14 The decay estimates for Einstein's equations

In this section we will establish the improved decay estimates for Einstein's equations. Our strategy is to use the weak decay estimates, obtained from the assumed energy bounds, to prove sharper decay estimates and then to recover the energy bounds in the next section.

Theorem 14.1. *Suppose that for some* $0 < \gamma \leq 1/2$

$$|\partial Z^I h| \leq C\varepsilon(1+t+|q|)^{-1/2-\gamma}(1+|q|)^{-1/2-\gamma}, \qquad |I| \leq N/2+4, \tag{14.1}$$

$$|Z^I h| \leq C\varepsilon(1+t)^{-1}, \qquad q=1, \qquad |I| \leq N/2+4 \tag{14.2}$$

hold for $0 \leq t \leq T$*. Then for* $0 \leq t \leq T$ *we have*

$$|Z^I h| \leq C\varepsilon(1+t+|q|)^{-1/2-\gamma}(1+|q|)^{1/2-\gamma}, \qquad |I| \leq N/2+4, \tag{14.3}$$

$$|\overline{\partial} Z^I h| \leq C\varepsilon(1+t+|q|)^{-3/2-\gamma}(1+|q|)^{1/2-\gamma}, \qquad |I| \leq N/2+3. \tag{14.4}$$

Assume also that h *satisfies the wave coordinate condition. Then for* $0 \leq t \leq T$ *we have*

$$|\partial h|_{L\mathcal{T}} + |\partial Z h|_{LL} \leq C\varepsilon(1+t)^{-1-2\gamma}, \qquad \text{and} \qquad |h|_{L\mathcal{T}} + |Zh|_{LL} \leq C\varepsilon(1+t)^{-1}(1+|q|). \tag{14.5}$$

Furthermore if in addition h *satisfies Einstein's equations then for* ε *sufficiently small and* $0 \leq t \leq T$ *we also have*

$$|\partial h|_{\mathcal{T}\mathcal{U}} \leq C\varepsilon(1+t)^{-1}, \qquad |h|_{\mathcal{T}\mathcal{U}} \leq C\varepsilon(1+t)^{-1}(1+|q|), \tag{14.6}$$

$$|\partial h| \leq C\varepsilon(1+t)^{-1}\ln(2+t). \tag{14.7}$$

In general, there are constants M_k*,* C_k *and* $\varepsilon_k > 0$ *such that if* $\varepsilon \leq \varepsilon_k$ *then for* $|I| = k \leq N/2+2$

$$|\partial Z^I h| \leq C_k\varepsilon(1+t)^{-1+M_k\varepsilon}, \qquad \text{and} \qquad |Z^I h| \leq C_k\varepsilon(1+t)^{-1+M_k\varepsilon}(1+|q|) \tag{14.8}$$

Remark 14.2. We remind the reader that, as stated in the Remark 2.4, our estimates make no distinction between the tensors h and $H = -h + O(h^2)$. In particular, one can directly verify that the conclusions of the theorem also hold for the tensor H.

First we note that all the estimates (14.3)-(14.8) trivially follow from the assumptions (14.1)-(14.2) away from the light cone, thus the theorem is only useful in the region $t/2 \leq |x| \leq 2t$. The estimate (14.3) follows from integrating (14.1) from $q = 1$, where (14.2) hold. Similarly the second parts of (14.5), (14.6) and (14.8) follow from integrating the first and using (14.2). It follows from (14.3) and Lemma 7.2 that we have the better estimate (14.4) for the derivatives tangential to the outgoing Minkowski cones. The inequalities (14.5)-(14.8) for tangential derivatives certainly follow from (14.4), so it only remains to prove these estimates for a derivative transversal to the light cone.
The missing improved estimates for a $(\partial_t - \partial_r)$ derivative transversal to the light cones will be obtained, in the case of (14.5), from the wave coordinate condition, see section 12, and for (14.6)-(14.8), from integrating the reduced Einstein wave equations, see section 10. The estimates from the wave coordinate condition are easily obtained. In fact the first estimate in (14.5) follows directly from Lemma 12.1 using the estimates (14.1), (14.3) and (14.4) and the second estimate in (14.5) follows integrating the first from $q = 1$ where (14.2) holds. However, the wave coordinate condition does not give estimates

for a transversal derivative of all components of the metric and the remaining components have to be controlled by integrating the wave equation expressed in polar coordinates. The estimates for the transversal derivative obtained from the wave coordinate condition rely on a decomposition of the metric with respect to the null frame. On the other hand, the estimates obtained from integrating the wave equation are based on a decomposition of the wave operator in terms of tangential derivatives and a transversal derivative.

14.1 Proof of (14.3) and (14.4)

For a fixed angular variable ω we integrate in the radial direction and use (14.1) and (14.2)

$$\begin{aligned}
(14.9)\quad |Z^I h(t,r,\omega)| &\leq |Z^I h(t,t+1,\omega)| + \int_r^{t+1} |\partial_r Z^I h(t,\rho,\omega)|\, d\rho \\
&\lesssim \frac{C\varepsilon}{1+t} + \int_r^{t+1} \frac{C\varepsilon\, d\rho}{(1+t+|t-\rho|)^{1/2+\gamma}(1+|t-\rho|)^{1/2+\gamma}} \lesssim \frac{C\varepsilon}{1+t} + \frac{C\varepsilon(1+|t-r|)^{1/2+\gamma}}{(1+t+r)^{1/2+\gamma}}
\end{aligned}$$

The estimate (14.3) now follows. By Lemma 7.2 and (14.3)

$$|\overline{\partial} Z^I h| \lesssim \frac{1}{1+t+|q|} \sum_{|J|\leq|I|+1} |Z^J h| \lesssim \frac{\varepsilon(1+|q|)^{1/2-\gamma}}{(1+t+|q|)^{3/2+\gamma}}$$

which proves (14.4).

14.2 Proof of (14.5).

We now show that the wave coordinate condition allows one to control certain components by lower order terms and terms with fast decay.

Lemma 14.3. *Suppose that the estimates* (14.1)- (14.4) *hold and that h satisfies the wave coordinate condition. Then*

$$(14.10)\qquad \sum_{|I|\leq k} |\partial Z^I h|_{LL} + \sum_{|J|\leq k-1} |\partial Z^J h|_{\mathcal{L}\mathcal{T}} \lesssim \sum_{|K|\leq k-2} |\partial Z^K h| + \varepsilon(1+t+|q|)^{-1-2\gamma}$$

Here the sum over $k-2$ is absent if $k\leq 1$ and the sum over $k-1$ is absent if $k=0$. Furthermore

$$(14.11)\qquad \frac{1}{1+|q|}\Big(\sum_{|I|\leq k} |Z^I h|_{LL} + \sum_{|J|\leq k-1} |Z^J h|_{\mathcal{L}\mathcal{T}} + \sum_{|K|\leq k-2} |Z^K h|\Big)(t,x) \lesssim \sup_{t/2\leq|y|\leq 3t/2} \sum_{|K|\leq k-2} |\partial Z^K h(t,y)| + \frac{\varepsilon}{1+t}$$

Proof. We first prove (14.10). Using the estimates of Lemma 12.2 derived from the wave coordinate condition followed by (14.1)- (14.4) we obtain

$$\begin{aligned}
(14.12)\quad \sum_{|I|\leq k} |\partial Z^I H|_{LL} + \sum_{|J|\leq k-1} |\partial Z^J H|_{LT} &\lesssim \sum_{|I|\leq k} |\bar\partial Z^I h| + \sum_{|K|\leq k-2} |\partial Z^K h| + \sum_{|I_1|+\ldots+|I_m|\leq k,\, m\geq 2} |Z^{I_m} h|\cdots|Z^{I_2} h|\, |\partial Z^{I_1} h| \\
&\leq \sum_{|K|\leq k-2} |\partial Z^K h| + \varepsilon(1+t+|q|)^{-1-2\gamma} + \varepsilon(1+t+|q|)^{-1-2\gamma}
\end{aligned}$$

The proof of estimate (14.11) for $|q| \geq t/2$ follows directly from (14.3). Thus we may assume that $|q| \leq t/2$. We now use the inequality

$$|H(t, r\omega)| \leq |H\big(t, (t+1)\omega\big)| + (1+|q|) \sup_{|\rho|\leq |q|+1} |\partial_\rho H(t, (t+\rho)\omega)|, \tag{14.13}$$

and the boundary condition (14.2) to conclude that

$$\frac{|Z^I H|_{LL} + |Z^J H|_{L\mathcal{T}} + |Z^K H|}{1+|q|} \lesssim \sup_{t/2\leq |y|\leq 2t} \Big(|\partial_r Z^I H|_{LL} + |\partial_r Z^J H|_{L\mathcal{T}} + |\partial_r Z^K H|\Big)(t,y) + \frac{\varepsilon}{1+t} \tag{14.14}$$

The desired result now follows from (14.10). □

The first part of (14.5) now follows directly from the Lemma with $k = 0, 1$ and the second part follows from integrating the first and using the boundary assumption (14.2) as in the proof of (14.3).

14.3 Proof of (14.6)-(14.7).

We will appeal to the L^∞ estimates of section 10 for the reduced wave equation

$$\widetilde{\square}_g h_{\mu\nu} = F_{\mu\nu},$$

where $F_{\mu\nu}$ is as in Lemma 3.2. We will now prove (14.6) and (14.7) assuming (14.1)-(14.5).

Lemma 14.4. *Suppose that the assumptions of Proposition 14.1 hold and let $F_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h)$ be as in Lemma 3.2. Then*

$$|F|_{\mathcal{T}\mathcal{U}} \leq C\varepsilon t^{-1-2\gamma} |\partial h| \tag{14.15}$$

and

$$|F| \leq C\varepsilon t^{-1-2\gamma} |\partial h| + C|\partial h|^2_{\mathcal{T}\mathcal{U}} \tag{14.16}$$

Proof. This follows from Lemma 13.3 using (14.1)-(14.5). □

Using the first part of Corollary 10.3; (10.24), and (14.1)-(14.5) and the previous lemma we get

Lemma 14.5. *With a constant depending on $\gamma > 0$ we have*

$$(1+t)\|\,|\partial h|_{\mathcal{T}\mathcal{U}}(t,\cdot)\|_{L^\infty} \leq C\varepsilon + C\varepsilon \int_0^t (1+\tau)^{-2\gamma} \|\partial h(\tau,\cdot)\|_{L^\infty}\, d\tau, \tag{14.17}$$

and

$$(1+t)\|\partial h(t,\cdot)\|_{L^\infty} \leq C\varepsilon + C\int_0^t \Big(\varepsilon(1+\tau)^{-2\gamma}\|\partial h(\tau,\cdot)\|_{L^\infty} + (1+\tau)\|\,|\partial h|_{\mathcal{T}\mathcal{U}}(\tau,\cdot)\|^2_{L^\infty}\Big)\, d\tau. \tag{14.18}$$

The estimates (14.6) and (14.7) now follow from the above lemma and the following technical result applied to $n_{00}(t) = (1+t)\|\,|\partial h|_{\mathcal{T}\mathcal{U}}(t,\cdot)\|_{L^\infty}$ and $n_{01}(t) = (1+t)\|\partial h(t,\cdot)\|_{L^\infty}$:

Lemma 14.6. *Suppose that* $n_{00} \geq 0$ *and* $n_{01} \geq 0$ *satisfy*

$$n_{00}(t) \leq C\varepsilon\Big(\int_0^t (1+s)^{-1-\gamma} n_{01}(s)\, ds + 1\Big) \tag{14.19}$$

$$n_{01}(t) \leq C\varepsilon\Big(\int_0^t (1+s)^{-1-\gamma} n_{01}(s)\, ds + 1\Big) + C\int_0^t (1+s)^{-1} n_{00}(s)^2\, ds \tag{14.20}$$

for some positive constants such that $0 < 16(C^2+C)\varepsilon < \gamma \leq 1$. *Then*

$$n_{00}(t) \leq 2C\varepsilon, \qquad \text{and} \qquad n_{01}(t) \leq 2C\varepsilon\big(1+\gamma \ln(1+t)\big) \tag{14.21}$$

Proof. Let T be the largest time such that

$$N_{01}(t) = \int_0^t (1+s)^{-1-\gamma} n_{01}(s)\, ds + 1 \leq 2, \qquad \text{for} \qquad t \leq T \tag{14.22}$$

Then for $t \leq T$ (14.21) holds and since,

$$\int_0^\infty (1+s)^{-1-\gamma}\big(1+\gamma\ln(1+s)\big)\, ds = \gamma^{-1}\int_0^\infty (1+\tau)\, e^{-\tau}\, d\tau = 2\gamma^{-1}+1$$

it follows that

$$N_{01}(t) \leq 2C\varepsilon\big(2\gamma^{-1}+1\big) + 1 \leq 3/2, \qquad \text{for} \qquad t \leq T.$$

Since $N_{01}(t)$ is continuous this contradicts that T is the maximal number such that (14.22) holds. Thus $T = \infty$ and (14.21) holds for all $t < \infty$. □

This proves the first part of (14.6) and (14.7). The second part of (14.6) follows from integrating the first using the boundary assumption (14.2) as in the proof of (14.3).

14.4 Proof of (14.8) in case $k = 1$.

We will now prove the first part of (14.8) for $|I| = 1$ assuming (14.1)-(14.7).

Lemma 14.7. *Suppose that the assumptions of Proposition 14.1 hold and let* $F_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h)$ *be as in Lemma 3.2. Then*

$$|ZF| \leq C\varepsilon t^{-1}\big(|\partial Z h| + |\partial h|\big) \tag{14.23}$$

Proof. This follows from Lemma 13.3. □

Using the second part of Corollary 10.3; (10.25), and (14.1)-(14.5) and the previous lemma we get

Lemma 14.8. *If* $\varepsilon > 0$ *is sufficiently small then*

$$(1+t)\sum_{|I|\leq 1} \|\partial Z^I h(t,\cdot)\|_{L^\infty} \leq C\varepsilon(1+t)^{C\varepsilon}\Big(1 + \int_0^t (1+\tau)^{-C\varepsilon} \sum_{|I|\leq 1} \|\partial Z^I h(\tau,\cdot)\|_{L^\infty}\, d\tau\Big). \tag{14.24}$$

The estimate (14.8) for $|I| = 1$ is now a consequence of the above lemma and the following technical result applied to $n_1(t) = (1+t)^{1-C\varepsilon}\sum_{|I|\leq 1}\|\partial Z^I h(t,\cdot)\|_{L^\infty}$:

Lemma 14.9. *Suppose that $n_1(t)\ge 0$ satisfies*

$$n_1(t)\le C\varepsilon\Big(1+\int_0^t (1+\tau)^{-1} n_1(\tau)\,d\tau\Big) \tag{14.25}$$

Then

$$n_1(t)\le C\varepsilon(1+t)^{C\varepsilon} \tag{14.26}$$

Proof.

$$N_1(t)=1+\int_0^t (1+\tau)^{-1} n_1(\tau)\,d\tau \tag{14.27}$$

satisfies $\dot N_1(t)\le C\varepsilon(1+\tau)^{-1}N_1(t)$. Multiplying by the integrating factor $(1+t)^{-C\varepsilon}$ and integrating we get $N_1(t)\le N_1(0)(1+t)^{C\varepsilon}=(1+t)^{C\varepsilon}$ and the lemma follows. $\square$

14.5 Proof of (14.8) in case $k\ge 1$.

We will now use induction to prove the first part of (14.8) for $|I|=k+1$ assuming that (14.1)-(14.5), the first part of (14.6), (14.7) and the first part of (14.8) for $|I|\le k$ hold.

Lemma 14.10. *Suppose that the assumptions of Proposition 14.1 hold and let $F_{\mu\nu}=F_{\mu\nu}(h)(\partial h,\partial h)$ be as in Lemma 3.2. Then*

$$|Z^I F|\le C\varepsilon t^{-1}\sum_{|K|\le|I|}|\partial Z^K h| + C\sum_{|J|+|K|\le|I|,\,|J|\le|K|<|I|}|\partial Z^J h||\partial Z^K h| \tag{14.28}$$

Proof. This follows from Lemma 13.3 using (14.1)-(14.7). $\square$

By Corollary 7.6

$$\begin{aligned}|\widetilde{\square}_g Z^I h| \lesssim |\hat Z^I F| &+ (1+t)^{-1}\sum_{|K|\le|I|,}\ \sum_{|J|+(|K|-1)_+\le|I|}|Z^J H||\partial Z^K h|\\ +C(1+q)^{-1}\sum_{|K|\le|I|}\Big(&\sum_{|J|+(|K|-1)_+\le|I|}|Z^J H|_{LL}+\sum_{|J'|+(|K|-1)_+\le|I|-1}|Z^{J'} H|_{L\mathcal{T}}+\sum_{|J''|+(|K|-1)_+\le|I|-2}|Z^{J''} H|\Big)|\partial Z^K h|\end{aligned} \tag{14.29}$$

where $(|K|-1)_+=|K|-1$, if $|K|\ge 1$, and 0, if $|K|=0$. Using Lemma 14.3 we get

$$(1+q)^{-1}\sum_{|J|\le k,\,|J'|\le k-1,\,|J''|\le k-2}|Z^J H|_{LL}+|Z^{J'} H|_{L\mathcal{T}}+|Z^{J''} H|\le \frac{C\varepsilon}{1+t}+\sum_{|J''|\le k-2}\sup_{t/2\le|y|\le 2t}|\partial Z^{J''} H(t,y)| \tag{14.30}$$

We hence obtain

$$|\widetilde{\square}_g Z^I h|\le C\varepsilon(1+t)^{-1}\sum_{|K|\le|I|}|\partial Z^K h| + \sum_{|J|+|K|\le|I|-1}\sup_{t/2\le|y|\le 2t}|\partial Z^J H(t,y)|\,|\partial Z^K h| \tag{14.31}$$

Then we have proven that

Lemma 14.11. *Let*

$$n_k(t) = (1+t) \sum_{|I|\le k} \|\partial Z^I h(t,\cdot)\|_{L^\infty}. \tag{14.32}$$

Then for $|I| = k$*:*

$$|\widetilde{\square}_g Z^I h| \le C(1+t)^{-2}\big(\varepsilon n_k(t) + n_{k-1}(t)^2\big) \tag{14.33}$$

By the first part of Corollary 10.3; (10.24), it therefore follows that:

Lemma 14.12.

$$n_k(t) \le C\varepsilon + C\int_0^t (1+\tau)^{-1}\big(\varepsilon n_k(\tau) + n_{k-1}(\tau)^2\big)\, d\tau \tag{14.34}$$

Our inductive hypothesis is $n_{k-1}(t)^2 \le C\varepsilon^2(1+t)^{C\varepsilon}$ so the bound $n_k(t) \le C\varepsilon(1+t)^{2C\varepsilon}$ follows from:

Lemma 14.13. *Suppose that*

$$n_k(t) \le C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon\int_0^t (1+\tau)^{-1} n_k(\tau)\, d\tau \tag{14.35}$$

then

$$n_k(t) \le C\varepsilon(1+t)^{2C\varepsilon}. \tag{14.36}$$

Proof. Let $N_k(t) = \int_0^t (1+\tau)^{-1} n_k(\tau)\, d\tau$. Then $|\dot N_k(t)| \le C\varepsilon(1+t)^{-1}\big((1+t)^{C\varepsilon} + N_k(t)\big)$. Multiplying by an integrating factor gives $\big(N_k(t)(1+t)^{-2C\varepsilon}\big)' \le C\varepsilon(1+t)^{-1-C\varepsilon}$ so $N_k(t)(1+t)^{-2C\varepsilon} \le C$ and hence $N_k(t) \le C(1+t)^{2C\varepsilon}$ and $n_k(t) \le 2C\varepsilon(1+t)^{2C\varepsilon}$. □

This proves the first part of (14.8). The second part of (14.8) follows from integrating the first and using the boundary assumption (14.2) as in the proof of (14.3).

15 Energy estimates for Einstein's equations

Recall the definitions

$$E_N(t) = \sup_{0\le\tau\le t} \sum_{|I|\le N} \int_{\Sigma_t} |\partial Z^I h|^2, \tag{15.1}$$

$$S_N(t) = \sum_{|I|\le N} \int_0^t \int_{\Sigma_\tau} \frac{\gamma\, |\bar\partial Z^I h|^2}{(1+|q|)^{1+2\gamma}} \tag{15.2}$$

In this section we prove the following theorem.

Theorem 15.1. *Assume that* $g = h + m$ *satisfies both Einstein's equations and the wave coordinate condition for* $0 \le t \le T$*. Suppose also that for some* $0 < \gamma \le 1/2$ *we have the following estimates for* $0 \le t \le T$*:*

1. *For all multi-indices $I,\ |I|\le N/2+4$*

$$|\partial Z^I h| + (1+|q|)^{-1}|Z^I h| + (1+t)(1+|q|)^{-1}|\bar\partial Z^I h| \le C\varepsilon(1+t)^{-1/2-\gamma}(1+|q|)^{-1/2-\gamma}, \tag{15.3}$$

2. *For all multi-indices $I,\ |I|\le N$*

$$|Z^I H(s,q,\omega)| \le C\varepsilon(1+t)^{-1}, \quad \textit{for } q=1, \tag{15.4}$$

3.

$$|\partial H|_{\mathcal{T}\mathcal{U}} + (1+|q|)^{-1}|H|_{\mathcal{T}\mathcal{U}} + (1+|q|)^{-1}|ZH|_{\mathcal{L}\mathcal{L}} \le C\varepsilon(1+t)^{-1}, \tag{15.5}$$

4. *For all multi-indices $I,\ |I|\le N/2+2$*

$$|\partial Z^I h| + (1+|q|)^{-1}|Z^I h| \le C\varepsilon(1+t)^{-1+C\varepsilon}, \tag{15.6}$$

5.

$$E_N(0)\le \varepsilon^2. \tag{15.7}$$

Then there are positive constants C_k independent of T such that if $\varepsilon\le C_k^{-2}$ we have the energy estimate

$$E_k(t)+S_k(t)\le 16\varepsilon^2(1+t)^{C_k\varepsilon}, \tag{15.8}$$

for $0\le t\le T$ and for all $k\le N$.

Remark 15.2. Once again we recall that our estimates hold simultaneously for the tensors h and $H=-h+O(h^2)$. We shall freely interchange h and H in the proof below.

Proof. Recall that the components of the tensor $h_{\mu\nu}=g_{\mu\nu}-m_{\mu\nu}$ satisfy the following wave equations:

$$\begin{aligned} &g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = F_{\mu\nu},, \\ &F_{\mu\nu} = P(\partial_\mu h,\partial_\nu h) + Q_{\mu\nu}(\partial h,\partial h) + G_{\mu\nu}(h)(\partial h,\partial h). \end{aligned} \tag{15.9}$$

where

$$P(\partial_\mu h,\partial_\nu h) = \frac14 m^{\alpha\alpha'}\partial_\mu h_{\alpha\alpha'}\, m^{\beta\beta'}\partial_\nu h_{\beta\beta'} - \frac12 m^{\alpha\alpha'}m^{\beta\beta'}\partial_\mu h_{\alpha\beta}\,\partial_\nu h_{\alpha'\beta'} \tag{15.10}$$

We prove the desired estimate by induction on k. We first establish the estimate

$$E_0(t)+S_0(t)\le 8\varepsilon^2(1+t)^{C_0\varepsilon} \tag{15.11}$$

for some constant C_0. After that we shall assume that the statement (15.8) for $k\le N'-1$ and prove the corresponding statement for $k\le N'$ with some constant $C_{N'}$. We shall base our argument on the energy estimate (11.3) for the solution of the wave equation $\widetilde{\Box}_g\phi=F$ proved in Proposition 11.1. Observe that the conditions of our Proposition on the tensor $h=g-m$ imply the assumptions of Proposition 11.1 for the metric g.

$$\int_{\Sigma_t}|\partial\phi|^2 + \int_0^t\int_{\Sigma_\tau}\frac{\gamma\,|\bar\partial\phi|^2}{(1+|q|)^{1+2\gamma}} \le 8\int_{\Sigma_0}|\partial\phi|^2 + C\varepsilon\int_0^t\int_{\Sigma_t}\frac{|\partial\phi|^2}{1+t} + 16\int_0^t\int_{\Sigma_t}|F|\,|\partial\phi| \tag{15.12}$$

15.1 The case of $N' = 0$.

In this section we prove the basic energy estimate for a solution of the equation (15.9).

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu} := P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h).$$

Recall that according to (13.10) of Lemma 13.3 we have a pointwise bound

$$|F| \lesssim |\partial h|^2_{\mathcal{T}\mathcal{U}} + |\bar\partial h||\partial h| + h|\partial h|^2$$

Using the assumptions of the proposition we infer that

$$|F| \lesssim \varepsilon \frac{|\partial h|}{1+t} \tag{15.13}$$

Therefore, the energy estimate (15.12) with $\phi = h_{\mu\nu}$ implies that

$$\int_{\Sigma_t} |\partial h|^2 + \int_0^t \int_{\Sigma_\tau} \frac{\gamma\, |\bar\partial h|^2}{(1+|q|)^{1+2\gamma}} \le 8 \int_{\Sigma_0} |\partial h|^2 + C_0 \varepsilon \int_0^t \int_{\Sigma_t} \frac{|\partial h|^2}{1+t}. \tag{15.14}$$

Using the smallness assumption on the initial data and the Gronwall inequality this, in turn, leads to the desired estimate (15.11).

$$E_0(t) + S_0(t) \le 8\varepsilon^2 (1+t)^{C_0\varepsilon}$$

15.2 The case of $N' = 1$.

To facilitate the exposition we first consider the case $N' = 1$. We start by noting that according to (7.18) of Corollary 7.6 we have that

$$\widetilde{\Box}_g Z h_{\mu\nu} = \hat{Z} F_{\mu\nu} + D_{\mu\nu},$$

where the term $D_{\mu\nu} = \widetilde{\Box}_g Z h_{\mu\nu} - \hat{Z}\widetilde{\Box}_g h_{\mu\nu}$ satisfies the estimate

$$|D| \lesssim \Big(\frac{|ZH| + |H|}{1+t} + \frac{|ZH|_{\mathcal{L}\mathcal{L}} + |H|_{\mathcal{L}\mathcal{T}}}{1+|q|} \Big) \sum_{|I|\le 1} |\partial Z^I h|$$

Recall that the tensor $H^{\alpha\beta} = -h^{\alpha\beta} + O(h^2)$. Thus using the assumptions on h of the proposition we derive that

$$|D| \lesssim \varepsilon \sum_{|I|\le 1} \frac{|\partial Z^I h|}{1+t}$$

On the other hand, inequality (13.11) gives the estimate

$$|ZF| \le \big(|\partial h|_{\mathcal{T}\mathcal{U}} + |\overline{\partial} h| + |h|\,|\partial h|\big)(|\partial Z h| + |\partial h|) + C|\partial h|\,|\overline{\partial} Z h| + C|\partial h|^2\,|Zh|$$

Using the assumptions of the proposition we conclude that

$$|\hat{Z} F| = |(Z + c_Z)F| \lesssim \varepsilon \sum_{|I|\le 1} \frac{|\partial Z^I h|}{1+t} + \varepsilon \frac{|\bar\partial Z h|}{(1+t)^{\frac12}(1+|q|)^{\frac12+\gamma}}$$

Now using the energy estimate (15.12) with $\phi = Zh_{\mu\nu}$ and $F = \hat{Z}F_{\mu\nu} + D_{\mu\nu}$ we obtain

$$\int_{\Sigma_t} |\partial Zh|^2 + \int_0^t \int_{\Sigma_\tau} \frac{\gamma\, |\bar\partial Zh|^2}{(1+|q|)^{1+2\gamma}} \le 8\int_{\Sigma_0} |\partial Zh|^2 + C\varepsilon \sum_{|I|\le 1} \int_0^t \int_{\Sigma_t} \frac{|\partial Z^I h|^2}{1+t} + C\varepsilon \int_0^t \int_{\Sigma_t} \frac{|\bar\partial Zh|\, |\partial Zh|}{t^{\frac12}(1+|q|)^{\frac12+\gamma}}$$
$$\le 8\int_{\Sigma_0} |\partial Zh|^2 + C\varepsilon \sum_{|I|\le 1} \int_0^t \int_{\Sigma_t} \frac{|\partial Z^I h|^2}{1+t} + C\varepsilon \int_0^t \int_{\Sigma_t} \frac{|\bar\partial Z^I h|^2}{(1+|q|)^{1+2\gamma}},$$

where we used the Cauchy-Schwarz inequality to pass to the last line. Combining this with the energy inequality (15.14) we infer that if $C\varepsilon \le \gamma/2$ then

$$\sum_{|I|\le 1} \int_{\Sigma_t} |\partial Z^I h|^2 + \sum_{|I|\le 1} \int_0^t \int_{\Sigma_\tau} \frac{\gamma\, |\bar\partial Z^I h|^2}{(1+|q|)^{1+2\gamma}} \le 16 \sum_{|I|\le 1} \int_{\Sigma_0} |\partial Z^I h|^2 + C_1\varepsilon \sum_{|I|\le 1} \int_0^t \int_{\Sigma_t} \frac{|\partial Z^I h|^2}{1+t} \tag{15.15}$$

The desired estimate

$$E_1(t) + S_1(t) \le 16\varepsilon^2 (1+t)^{C_1\varepsilon}$$

now follows from the Gronwall inequality and the smallness assumption on the initial data.

15.3 The case of $N' > 1$.

In what follows we assume that we have already shown that

$$E_{N'-1}(t) + S_{N'-1}(t) \le 16\varepsilon^2 (1+t)^{C_{N'-1}\varepsilon}, \tag{15.16}$$

and prove that there exists a constant $C_{N'}$ such that

$$E_{N'}(t) + S_{N'}(t) \le 16\varepsilon^2 (1+t)^{C_{N'}\varepsilon}, \tag{15.17}$$

We start this section by writing the wave equation for the quantity $Z^I h_{\mu\nu}$ with $|I| = N'$

$$\widetilde{\Box}_g Z^I h_{\mu\nu} = \hat{Z}^I F_{\mu\nu} + D^I_{\mu\nu},$$

where

$$D^I_{\mu\nu} = \widetilde{\Box}_g Z^I h_{\mu\nu} - \hat{Z}^I \widetilde{\Box}_g h_{\mu\nu}$$

We apply the energy estimate (15.12) with the functions $\phi = Z^I h_{\mu\nu}$ and $F = \hat{Z}^I F_{\mu\nu} + D^I_{\mu\nu}$

$$\int_{\Sigma_t} |\partial Z^I h|^2 + \int_0^t \int_{\Sigma_\tau} \frac{\gamma\, |\bar\partial Z^I h|^2}{(1+|q|)^{1+2\gamma}} \le 8\int_{\Sigma_0} |\partial Z^I h|^2 + C\varepsilon \int_0^t \int_{\Sigma_t} \frac{|\partial Z^I h|^2}{1+t} + 16 \int_0^t \int_{\Sigma_t} \big(|\hat{Z}^I F| + |D^I|\big)\, |\partial Z^I h| \tag{15.18}$$

Note that we can estimate

$$\int_0^t \int \big(|\hat{Z}^I F| + |D^I|\big)\, |\partial Z^I h|\, dx\, dt \lesssim \int_0^t \frac{\varepsilon}{1+t} |\partial Z^I h|^2\, dx\, dt + \int_0^t \int \varepsilon^{-1}(1+t)\big(|\hat{Z}^I F|^2 + |D^I|^2\big)\, dx\, dt \tag{15.19}$$

Here the first term is of the type that appears already in the energy estimate (15.18). Thus it remains to handle the second term.

According to (7.21) of Corollary 7.6 we have that

$$D^I = \sum_{k=0}^{|I|} D_k^I, \tag{15.20}$$

$$D_k^I = D_k^{I1} + D_k^{I2} + D_k^{I3} + D_k^{I4}, \tag{15.21}$$

$$|D_k^{I1}| \lesssim \sum_{|K|=k} \sum_{|J|+(|K|-1)_+\leq |I|} \frac{|Z^J H|}{1+t+|q|}\, |\partial Z^K h|, \tag{15.22}$$

$$|D_k^{I2}| \lesssim \sum_{|K|=k} \sum_{|J|+(|K|-1)_+\leq |I|} \frac{|Z^J H|_{LL}}{1+|q|}\, |\partial Z^K h|, \tag{15.23}$$

$$|D_k^{I3}| \lesssim \sum_{|K|=k} \sum_{|J|+(|K|-1)_+\leq |I|-1} \frac{|Z^J H|_{L\mathcal{T}}}{1+|q|}\, |\partial Z^K h|, \tag{15.24}$$

$$|D_k^{I4}| \lesssim \sum_{|K|=k} \sum_{|J|+(|K|-1)_+\leq |I|-2} \frac{|Z^J H|}{1+|q|}\, |\partial Z^K h|, \tag{15.25}$$

The estimates for D_k^I with $k \leq N/2$. We must now estimate

$$\int_0^t \int \varepsilon^{-1}(1+t)|D_k^I|^2\, dx\, dt \tag{15.26}$$

Since $k = |K| \leq N/2$ in (15.22)-(15.25) it follows from the assumptions in the theorem that we can estimate

$$\varepsilon^{-1}(1+t)|\partial Z^K h|^2 \lesssim \min\Big(\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}, \frac{\varepsilon}{(1+|q|)^{1+2\gamma}}\Big) \tag{15.27}$$

and it thus suffices to estimate

$$\int_0^t \int \varepsilon^{-1}(1+t)|D_k^{I1}|^2\, dx\, dt \lesssim \sum_{|J|\leq |I|} \int_0^t \int \frac{\varepsilon}{(1+|q|)^{1+2\gamma}} \frac{|Z^J H|^2}{(1+t+|q|)^2}\, dx\, dt, \tag{15.28}$$

$$\int_0^t \int \varepsilon^{-1}(1+t)\big(|D_k^{I3}|^2 + |D_k^{I4}|^2\big)\, dx\, dt \lesssim \sum_{|J|\leq |I|-1} \int_0^t \int \frac{\varepsilon}{(1+t)^{1-C\varepsilon}} \frac{|Z^J H|^2}{(1+|q|)^2}\, dx\, dt, \tag{15.29}$$

$$\int_0^t \int \varepsilon^{-1}(1+t)|D_k^{I2}|^2\, dx\, dt \lesssim \sum_{|J|\leq |I|} \int_0^t \int \min\Big(\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}, \frac{\varepsilon}{(1+|q|)^{1+2\gamma}}\Big) \frac{|Z^J H|^2_{\mathcal{L}\mathcal{L}}}{(1+|q|)^2}\, dx\, dt, \tag{15.30}$$

Lemma 15.3. *Let f be a smooth function satisfying the condition*

$$|f| \lesssim \varepsilon(1+t)^{-1}, \qquad \textit{for} \qquad q = 1 \tag{15.31}$$

Then

$$\int_0^t \int \frac{\varepsilon}{(1+|q|)^{1+2\gamma}} \frac{|f|^2}{(1+t+|q|)^2}\, dx\, dt \lesssim \int_0^t \frac{\varepsilon}{(1+t)^{1+2\gamma}} \int |\partial f|^2\, dx\, dt + \varepsilon^3 \tag{15.32}$$

and

$$\int_0^t \int \frac{\varepsilon}{(1+t)^{1-C\varepsilon}} \frac{|f|^2}{(1+|q|)^2}\, dx\, dt \lesssim \int_0^t \frac{\varepsilon}{(1+t)^{1-C\varepsilon}} \Big(\varepsilon^2 + \int |\partial f|^2\, dx\Big)\, dt \tag{15.33}$$

Furthermore,

$$\begin{aligned} \int_0^t \int \min\Big(\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}, \frac{\varepsilon}{(1+|q|)^{1+2\gamma}}\Big) \frac{|f|^2}{(1+|q|)^2}\, dx\, dt \lesssim \int_0^t \int \frac{\varepsilon|\partial_r f|^2}{(1+|q|)^{1+2\gamma}}\, dx dt \\ + \varepsilon^2 \int_0^t \frac{\varepsilon}{(1+t)^{1-C\varepsilon}}\, dt \end{aligned} \tag{15.34}$$

Proof. We shall repeatedly use the Poincaré inequality (9.1) of Lemma 9.1

$$\int\limits_{\Sigma_t} \frac{|f(x)|^2\, dx}{(1+|q|)^{2+2\sigma}} \lesssim \int\limits_{S_{(t+1)}} |f|^2\, dS + \int\limits_{\Sigma_t} \frac{|\partial_r f(x)|^2\, dx}{(1+|q|)^{2\sigma}} \tag{15.35}$$

which holds for any value of $\sigma > -1/2$, $\sigma \neq 1/2$. In particular, using (15.31), we obtain that

$$\int\limits_{\Sigma_t} \frac{|f|^2\, dx}{(1+|q|)^{2+2\sigma}} \lesssim \varepsilon^2 + \int\limits_{\Sigma_t} \frac{|f|^2\, dx}{(1+|q|)^{2\sigma}} \tag{15.36}$$

The estimates (15.32) and (15.33) now follow from (15.36) with $\sigma = 0$. $\square$

We now note the following generalization of (15.35)

$$\int\limits_{\Sigma_t} \min\Big(\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}, \frac{\varepsilon}{(1+|q|)^{1+2\gamma}}\Big) \frac{|f(x)|^2\, dx}{(1+|q|)^2} \lesssim \frac{\varepsilon}{(1+t)^{1-C\varepsilon}} \int\limits_{S_{(t+1)}} |f|^2\, dS + \varepsilon \int\limits_{\Sigma_t} \frac{|\partial_r f(x)|^2\, dx}{(1+|q|)^{1+2\gamma}} \tag{15.37}$$

The proof of (15.37) can be reduced to (15.35) by subtracting a term which picks up the boundary value. We define

$$\widetilde{f} = f - \overline{f}, \qquad \text{where} \qquad \overline{f}(r,\omega) = f\big((t+1),\omega\big)\chi(r/t) \tag{15.38}$$

and $\chi(s) = 1$, when $3/4 \le s \le 3/2$ and $\chi(s) = 0$ when $s \le 1/2$ or $s \ge 2$. Then

$$\int\limits_{\Sigma_t} \min\Big(\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}, \frac{\varepsilon}{(1+|q|)^{1+2\gamma}}\Big) \frac{|f(x)|^2\, dx}{(1+|q|)^2} \lesssim \varepsilon \int\limits_{\Sigma_t} \frac{|\tilde{f}(x)|^2\, dx}{(1+|q|)^{3+2\gamma}} + \frac{\varepsilon}{(1+t)^{1-C\varepsilon}} \int\limits_{\Sigma_t} \frac{|\bar{f}(x)|^2\, dx}{(1+|q|)^2} \tag{15.39}$$

We now apply (15.35) to the function $\tilde{f}$, which vanishes at $r = t+1$, and observe that

$$\int\limits_{\Sigma_t} \frac{|\partial_r \overline{f}(x)|^2\, dx}{(1+|q|)^{1+2\gamma}} \lesssim \int f\big((t+1),\omega\big)^2 d\omega \lesssim \frac{1}{(1+t)^2} \int_{S_{t+1}} |f|^2\, dS \tag{15.40}$$

On the other hand,

$$\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}\int_{\Sigma_t}\frac{|\overline{f}(x)|^2\,dx}{(1+|q|)^2}\lesssim\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}\int_{S_{t+1}}|f|^2\,dS \tag{15.41}$$

which proves (15.37).

Using the lemma above with $f=Z^JH$, together with (15.28), (15.29) and the assumption that $E_{N'-1}\le 16(1+t)^{C_{N'-1}\varepsilon}$ we see that we can estimate

$$\begin{aligned}\int_0^t\int\varepsilon^{-1}(1+t)\big(|D_k^{I1}|^2+|D_k^{I3}|^2+|D_k^{I4}|^2\big)\,dx\,dt&\lesssim\int_0^t\frac{\varepsilon}{(1+t)^{1+2\gamma}}E_{N'}(t)\,dt+\varepsilon^2\int_0^t\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}\,dt\\&\lesssim\varepsilon\,E_{N'}(t)+\varepsilon^2\int_0^t\frac{\varepsilon}{(1+t)^{1-C\varepsilon}}\,dt\end{aligned}$$

for all $\le N/2$.

It thus remains the term (15.30) containing D_k^{I2}. We shall use the version of the Poincaré inequality (15.34) to create the term $\partial_q(Z^JH)_{LL}$, which can be then converted to a tangential derivative of Z^JH via the wave coordinate condition. However, in order to implement this strategy we modify the term Z^JH_{LL} according to Lemma 12.2. We recall the notation

$$H^J_{\mu\nu}=Z^JH_{\mu\nu}+\sum_{|J'|<|J|}c_{J'}{}^{J}{}_{\mu}{}^{\gamma}\,Z^JH_{\gamma\nu} \tag{15.42}$$

If $|J|\le N'$ then the lower order terms in the right hand side of (15.42) may be estimated using (15.29) and (15.33) as before. According to Lemma 12.2 and the pointwise estimates in (15.6) and (15.4)

$$\begin{aligned}|\partial_rH^J_{LL}|&\lesssim\sum_{|J'|\le|J|}|\bar\partial Z^{J'}H|+\sum_{|J_1|+..+|J_m|\le|J|,\ m\ge2}|Z^{J_m}H|\cdots|Z^{J_2}H|\,|\partial Z^{J_1}H|\\&\lesssim\sum_{|J'|\le|J|}|\bar\partial Z^{J'}H|+\sum_{|J_1|+|J_2|\le|J|}|Z^{J_1}H|\,|\partial Z^{J_2}H|\\&\lesssim\sum_{|J'|\le|J|}|\bar\partial Z^{J'}H|+\frac{\varepsilon(1+|q|)^{1/2-\gamma}}{(1+t)^{1/2+\gamma}}|\partial Z^{J'}H|+\frac{\varepsilon|Z^{J'}H|}{(1+t)^{1/2+\gamma}(1+|q|)^{1/2+\gamma}}\end{aligned} \tag{15.43}$$

Hence

$$\int_0^t\int\frac{\varepsilon|\partial_rH^J_{LL}|^2}{(1+|q|)^{1+2\gamma}}\,dxdt\lesssim\sum_{|J'|\le|J|}\int_0^t\int\Big(\frac{\varepsilon|\overline{\partial}Z^{J'}H|^2}{(1+|q|)^{1+2\gamma}}+\frac{\varepsilon|\partial Z^{J'}H|^2}{(1+t)^{1+2\gamma}}+\frac{\varepsilon|Z^{J'}H|^2}{(1+t)^{1+2\gamma}(1+|q|)^2}\Big)\,dxdt \tag{15.44}$$

If we use (15.33) with $C\varepsilon$ in the exponent replaced by 2γ we see that the last term can be estimated by the second term from the right plus a term from the boundary:

$$\int_0^t\int\frac{\varepsilon|\partial_rH^J_{LL}|^2}{(1+|q|)^{1+2\gamma}}\,dxdt\lesssim\sum_{|J'|\le|J|}\int_0^t\int\Big(\frac{\varepsilon|\overline{\partial}Z^{J'}H|^2}{(1+|q|)^{1+2\gamma}}+\frac{\varepsilon|\partial Z^{J'}H|^2}{(1+t)^{1+2\gamma}}+\frac{\varepsilon}{(1+t)^{1+2\gamma}}\varepsilon^2\Big)\,dxdt \tag{15.45}$$

As we argued, when estimating (15.30) we can replace $|Z^J H|_{LL}$ by the left hand side of (15.42). After that we use the version of the Poincare inequality (15.34) applied to H^J_{LL} and this together with (15.45) gives

$$\int_0^t\int \varepsilon^{-1}(1+t)|D_k^{I2}|^2\,dxdt \lesssim \varepsilon S_{N'}(t)+\varepsilon E_{N'}(t)+\varepsilon^2\int_0^t \frac{\varepsilon}{(1+t)^{1-C\varepsilon}}\,dt \tag{15.46}$$

Summarizing, we have proven that

$$\int_0^t\int \varepsilon^{-1}(1+t)|D_k^{I}|^2\,dxdt \lesssim \varepsilon S_{N'}(t)+\varepsilon E_{N'}(t)+\varepsilon^2\int_0^t \frac{\varepsilon}{(1+t)^{1-C\varepsilon}}\,dt, \qquad k\le N/2 \tag{15.47}$$

This concludes the estimates in the case $k \le N/2$.

The commutator in case $k \ge N/2$. We isolate the case when $|K| = N' = |I|$. We can estimate its contribution to the $D^I_{N'}$ by the following expression:

$$|D^I_{N'}| \lesssim \sum_{|K|=|I|}\Big(\frac{|H|+|ZH|}{1+t+|q|}+\frac{|ZH|_{\mathcal{L}\mathcal{L}}+|H|_{\mathcal{L}\mathcal{T}}}{1+|q|}\Big)|\partial Z^K h| \lesssim \varepsilon\sum_{|K|=|I|}\frac{|\partial Z^K h|}{1+t},$$

where to pass to the last line we used pointwise estimates from (15.5), (15.3), and (15.4). In the case when $N/2 \le k < |I|$ we estimate the contribution of the corresponding term in D^I_k, with the help of (15.6) as follows:

$$|D^I_k| \lesssim \sum_{|K|<|I|}\sum_{|J|\le N/2}\frac{|ZH|}{1+|q|}|\partial Z^K h| \lesssim \varepsilon\sum_{|K|<|I|}\frac{|\partial Z^K h|}{(1+\tau)^{1-C\varepsilon}}$$

Therefore,

$$\int_0^t\int_{\Sigma_\tau} \varepsilon^{-1}(1+t)|D_k^{I}|^2\,dxdt \lesssim \varepsilon\int_0^t\int \sum_{|K|<|I|}\frac{|\partial Z^K h|^2}{(1+\tau)^{1-2C\varepsilon}}+\sum_{|K|=|I|}\frac{|\partial Z^K h|^2}{1+\tau}\,dxdt \tag{15.48}$$

Using the inductive assumption (15.16) we can therefore estimate

$$\int_0^t\int_{\Sigma_\tau} \varepsilon^{-1}(1+t)|D_k^{I}|^2\,dxdt \lesssim \varepsilon\int_0^t \frac{E_{N'}(\tau)}{1+\tau}\,dt+\varepsilon^2\int_0^t\frac{\varepsilon\,dt}{(1+\tau)^{1-2C\varepsilon}}, \qquad N/2\le k\le N' \tag{15.49}$$

The inhomogeneous term. By (13.12)

$$\begin{aligned} |\hat{Z}^I F| \lesssim &\sum_{|J|+|K|\le|I|}\big(|\partial Z^J h|_{\mathcal{T}\mathcal{U}}\,|\partial Z^K h|_{\mathcal{T}\mathcal{U}}+|\bar\partial Z^J h|\,|\partial Z^K h|\big)+\sum_{|J|+|K|\le|I|-1}|\partial Z^J h|_{L\mathcal{T}}|\partial Z^K h|\\ &+\sum_{|J|+|K|\le|I|-2}|\partial Z^J h|\,|\partial Z^K h|+\sum_{|J_1|+\ldots+|J_m|\le|I|,\,m\ge3}|Z^{J_m}h|\cdots|Z^{J_3}h|\,|\partial Z^{J_2}h||\partial Z^{J_1}h| \end{aligned} \tag{15.50}$$

The highest order terms with one of $|J|$, $|K|$ or $|I_i|$ equal to $N=|I|$ are bounded by

$$\begin{aligned} &\big(|\partial h|_{\mathcal{T}\mathcal{U}}+|\bar\partial h|+|h||\partial h|\big)\sum_{|I|=N}|\partial Z^I h|+|\partial h|^2\sum_{|I|=N}|Z^I h|+|\partial h|\sum_{|I|=N}|\bar\partial Z^I h|\\ &\le\frac{\varepsilon}{1+t}\sum_{|I|=N}|\partial Z^I h|+\frac{\varepsilon^2}{(1+t)^{1+2\gamma}(1+q)^{1+2\gamma}}\sum_{|I|=N}|Z^I h|+\frac{\varepsilon}{(1+t)^{1/2+\gamma}(1+q)^{1/2+\gamma}}\sum_{|I|=N}|\bar\partial Z^I h| \end{aligned} \tag{15.51}$$

The remaining lower order terms are of the form

$$\sum_{|K|<N,|J|\le N/2} |\partial Z^J h|\,|\partial Z^K h| + \sum_{|K|<N,\,|J|,|L|\le N/2} |\partial Z^J h|\,|\partial Z^L h|\,|Z^K h| \tag{15.52}$$
$$\le \frac{\varepsilon}{(1+t)^{1-C\varepsilon}} \sum_{|K|<N} |\partial Z^K h| + \frac{\varepsilon^2}{(1+t)^{1+2\gamma}(1+q)^{1+2\gamma}} \sum_{|I|<N} |Z^I h|.$$

It therefore follows that

(15.53)
$$\int_0^t \int_{\Sigma_\tau} \varepsilon^{-1}(1+t)|\hat{Z}^I F|^2\, dxdt \lesssim \sum_{|K|\le|I|} \varepsilon \int_0^t \int \frac{|\partial Z^K h|^2}{1+\tau} + \frac{|\overline{\partial} Z^K h|^2}{(1+|q|)^{1+2\gamma}} + \frac{|Z^K h|^2}{(1+t)^{1+2\gamma}(1+|q|)^2}\, dxdt$$
$$+ \int_0^t \frac{\varepsilon\, dt}{(1+\tau)^{1-2C\varepsilon}} \sum_{|I|<N} |\partial Z^I h|^2\, dxdt$$
$$\lesssim \sum_{|K|\le|I|} \varepsilon \int_0^t \int \frac{|\partial Z^K h|^2}{1+\tau} + \frac{|\overline{\partial} Z^K h|^2}{(1+|q|)^{1+2\gamma}} + \frac{\varepsilon^2}{(1+t)^{1+2\gamma}}\, dxdt + \int_0^t \frac{\varepsilon\, dt}{(1+\tau)^{1-2C\varepsilon}} \sum_{|I|<N} |\partial Z^I h|^2\, dxdt$$

Here, to estimate the last term in the first row we used (15.33) with $-C\varepsilon$ in the exponent replaced by 2γ, which produced a term similar to the first term of the first line plus a boundary term. Using the inductive assumption (15.16) we thus obtain

$$\int_0^t \int_{\Sigma_\tau} \varepsilon^{-1}(1+t)|\hat{Z}^I F|^2\, dxdt \lesssim \varepsilon \int_0^t \frac{E_{N'}(\tau)\, d\tau}{1+\tau} + \varepsilon S_{N'}(t) + \varepsilon^2 \int_0^t \frac{\varepsilon\, d\tau}{(1+\tau)^{1-C\varepsilon}} \tag{15.54}$$

The conclusion of the proof in case $N' > 1$ The inequalities (15.18)-(15.19) and (15.47), (15.49) and (15.54) imply that for some constant C:

$$E_{N'}(t) + S_{N'}(t) \le 8E_{N'}(0) + C\varepsilon\big(E_{N'}(t) + S_{N'}(t)\big) + C\varepsilon \int_0^t \frac{E_{N'}(\tau)\, d\tau}{1+\tau} + C\varepsilon^2 \int_0^t \frac{\varepsilon\, d\tau}{(1+\tau)^{1-C\varepsilon}} \tag{15.55}$$

If we now choose ε so small that $C\varepsilon \le 1/9$ we can move the second term on the right to the left and multiply by $9/8$ to obtain for some new constants

$$E_{N'}(t) + S_{N'}(t) \le 9E_{N'}(0) + C\varepsilon \int_0^t \frac{E_{N'}(\tau)\, d\tau}{1+\tau} + C\varepsilon^2 \int_0^t \frac{\varepsilon\, d\tau}{(1+\tau)^{1-C\varepsilon}} \tag{15.56}$$

This can now be integrated using a Grönwall type of argument. If $G(t)$ denotes the right hand side then we have

$$G'(t) \le \frac{C\varepsilon}{1+t} G(t) + \frac{C\varepsilon^3}{(1+t)^{1-C\varepsilon}}$$

Multiplying with the integrating factor we get

$$\frac{d}{dt}\Big(G(t)(1+t)^{-C\varepsilon}\Big) \le \frac{C\varepsilon^3}{1+t}$$

and hence if integrate and use that $C\varepsilon \ln(1+t) \le (1+t)^{C\varepsilon}$, for $t \ge 0$ (as is seen by differentiating both sides), and use that by assumption (15.7) $G(0) \le 9\varepsilon^2$, we obtain

$$G(t) \le G(0)(1+t)^{C\varepsilon} + C\varepsilon^3 \ln(1+t)(1+t)^{C\varepsilon} \le 9\varepsilon^2(1+t)^{C\varepsilon} + \varepsilon^2(1+t)^{2C\varepsilon} \le 10\varepsilon^2(1+t)^{2C\varepsilon}$$

Hence we have proven that

$$E_{N'}(t) + S_{N'}(t) \le 10\varepsilon^2(1+t)^{2C\varepsilon}$$

This concludes the induction and the proof of the theorem. □

16 Geodesic completeness

Having constructed a solution metric $g = m + h$ of the Einstein equations we need to verify that the resulting space-time $(\mathbb{R}^4, g)$ is causally geodesically complete. Let

$$X(\tau) = (x^0(\tau), x(\tau)) = (t(\tau), x(\tau) = (t(\tau), r\omega(\tau))$$

be a causal geodesic parameterized by the affine parameter τ. Such geodesics satisfy the equations

$$\begin{aligned} &\ddot{X}^\alpha(\tau) + \Gamma^\alpha_{\beta\gamma}(X(\tau))\dot{X}^\beta \dot{X}^\gamma = 0, \\ &X(0) = Y, \quad \dot{X}(0) = \xi \end{aligned} \tag{16.1}$$

where Y is the point of the origin of the geodesic $X(\tau)$ and ξ is the initial velocity satisfying the condition

$$g_{\alpha\beta}(Y)\xi^\alpha \xi^\beta = -A^2 \le 0 \tag{16.2}$$

for some constant A. Condition (16.2) is preserved in time, i.e.,

$$g_{\alpha\beta}(X(\tau))\dot{X}^\alpha \dot{X}^\beta = -A^2 \tag{16.3}$$

In the following lemma we show that a vector η causal with respect to the metric g is "almost" causal with respect to the Minkowski metric m.

Lemma 16.1. *Let η be a causal 4-vector, i.e,*

$$g_{\alpha\beta}\eta^\alpha \eta^\beta \le -A^2 \le 0 \tag{16.4}$$

for some non-negative constant A. Then

$$A + |\eta^i| \le 2|\eta^0|, \qquad \forall i = 1, .., 3 \tag{16.5}$$

Proof. Expanding $g = m + h$ we obtain from (16.4) that

$$-|\eta^0|^2 + \sum_{i=1}^3 |\eta^i|^2 \le |h| \cdot (|\eta^0|^2 + \sum_{i=1}^3 |\eta^i|^2)$$

and the desired estimate follows provided that $|h| \le 1/4$. □

We choose a future oriented initial velocity ξ, i.e., $\dot x^0(0)>0$.

Proposition 16.2. *Assume that $h=g-m$ satisfies the estimates*[14]

$$|h||\partial h|+|\partial h|_{TU}+|\bar\partial h|_{\underline{LL}}\lesssim \varepsilon t^{-1},$$
$$|\partial h(t,x)|\lesssim \varepsilon t^{-1},\quad for\quad |x|\le t/2$$

Let $X(\tau)$ is a future inextendible causal geodesic. Then the values of the affine parameter τ span the interval $[0,\infty)$.

Proof. We start by considering a time-like geodesic $X(\tau)$. Reparameterizing, if necessary, we can assume that the constant $A=1$ in (16.3). Then equation (16.3) and inequality (16.5) with $A=1$ imply that for all $\tau\ge 0$.

$$\dot x^0(\tau)\ge \frac12+|\dot x(\tau)| \tag{16.6}$$

We removed the absolute value from $\dot x^0(\tau)$, since $\dot x^0(0)>0$. This is the only part of the argument, which uses the fact that $X(\tau)$ is a time-like geodesic. The case of a null geodesic will require an additional argument.

Assume that $X(\tau)$ is a time-like geodesic of finite length τ_*. We first observe that

$$\lim_{\tau\to\tau_*}|X(\tau)|=\infty$$

which means that $X(\tau)$ escapes to infinity[15] in finite proper time τ_*. This easily follows from the standard ODE theory. The inequality (16.6) implies that $\dot x^0(\tau)$ controls $\dot X(\tau)$. Thus to obtain contradiction it suffices to show that

$$\lim_{\tau\to\tau_*}x^0(\tau)<\infty$$

Throughout this section we will use consistently use the notation $x^0=t$. We recall that

$$\Gamma^\alpha_{\beta\gamma}=g^{\alpha\sigma}(\partial_\beta h_{\gamma\sigma}+\partial_\gamma h_{\beta\sigma}-\partial_\sigma h_{\beta\gamma})$$

Thus, expanding the metric $g=m+h$,

$$\ddot x^0-(2\partial_\beta h_{0\gamma}-\partial_0 h_{\beta\gamma})\dot x^\beta\dot x^\gamma=h\cdot\partial h\cdot|\dot X|^2$$

We further observe that

$$\partial_\beta h_{0\gamma}\dot x^\beta\dot x^\gamma=\frac{d}{d\tau}\big(h_{0\gamma}\dot x^\gamma\big)-h_{0\gamma}\ddot x^\gamma \tag{16.7}$$

We now additionally recall that $\partial_q h_{\underline{LL}}$ is the only derivative of h that does not have the decay rate of at least $(x^0)^{-1}$. Thus

$$\partial_0 h_{\beta\gamma}\dot X^\beta\dot X^\gamma=\partial_q h_{\beta\gamma}\dot X^\beta\dot X^\gamma+\varepsilon O(t^{-1})|\dot X|^2=\partial_q h_{\underline{LL}}|\dot X^{\underline{L}}|^2+\varepsilon O((x^0)^{-1})|\dot X|^2$$

[14]These assumptions are consistent with the decay estimates for h proved in Theorem 14.1.
[15]viewed from the point of view of the global system of wave coordinates on $\mathbb{R}^4$.

The expression

$$\dot{X}^{\underline{L}} = \dot{X}^\alpha L_\alpha = -\dot{x}^0 + \frac{x^i}{|x|}\dot{x}^i = -\frac{d}{d\tau}(t-r) = -\dot{q}$$

Moreover,

$$\partial_q h_{\underline{LL}} = 4\partial_q h_{00} + \varepsilon O((x^0)^{-1})$$

Furthermore, introduce $\zeta(x^0/r)$ a cut-off function of the set $r \geq x^0/2$. Then

$$\partial_q h_{00} = (1-\zeta)\partial_q h_{00} + \zeta\partial_q h_{00} = \varepsilon O(t^{-1}) + \partial_q(\zeta h_{00}) - (\partial_q \zeta(x^0/r))h_{00}$$

We compute

$$\partial_q \zeta(x^0/r) = (r^{-1} + x^0 r^{-2})\zeta'(x^0/r) \lesssim (x^0)^{-1}$$

since $r \geq x^0/2$ on the support of $\zeta'(x^0/r)$. Thus $\partial_q h_{00}$ can be replaced by $\partial_q(\zeta h_{00})$ at the expense of a term of order $\varepsilon O((x^0)^{-1})$. Therefore,

$$\begin{aligned}\partial_q h_{\underline{LL}}|\dot{X}^{\underline{L}}|^2 &= 4\partial_q h_{00}|\dot{q}|^2 = 4\partial_q(\zeta h_{00})|\dot{q}|^2 + \varepsilon O((x^0)^{-1})|\dot{q}|^2\\ &= 4\frac{d}{d\tau}\big(\zeta h_{00}\,\dot{q}\big) - 4\zeta h_{00}\ddot{q} - 4\partial_L(\zeta h_{00})\dot{X}^L\dot{X}^{\underline{L}} - 4\partial_\Omega(\zeta h_{00})\dot{X}^\omega \dot{X}^{\underline{L}} + \varepsilon O((x^0)^{-1})\end{aligned}$$

Here,

$$h(X(\tau)) = h(q((\tau), v(\tau), \omega(\tau))$$

where $q = x^0 - r$, $v = x^0 + r$, and $\omega = \frac{x^i}{r}$. The advantage is that $\partial_\Omega h_{00}, \partial_L h_{00}$ already decay faster than $(x^0)^{-1}$ and $\partial_\Omega \zeta(x^0/r) = 0$, while $|\partial_L \zeta(x^0/r)| \lesssim (x^0)^{-1}$. Thus

$$\partial_q h_{\underline{LL}}|\dot{X}^{\underline{L}}|^2 = \frac{d}{d\tau}\big(\zeta h_{00}\dot{q}\big) - \zeta h_{00}\ddot{q} + \varepsilon O((x^0)^{-1})$$

It remains to analyze the term

$$\ddot{q} = \frac{d}{d\tau}(\dot{x}^0 - \frac{x^i}{r}\dot{x}^i) = \ddot{x}^0 - \ddot{x}^i\frac{x^i}{r} + r^{-1}(|\dot{x}|^2 - r^{-2}|x\cdot\dot{x}|^2) \tag{16.8}$$

From the geodesic equation (16.1) we can estimate

$$|\ddot{x}^\alpha| \leq |\partial h||\dot{X}|^2$$

Additionally, since on the support of $\zeta(x^0/r)$, $r \geq x^0/2$, we have that the last term in (16.8) multiplied by ζh_{00} contributes at most[16] $\varepsilon(x^0)^{-1}$. Thus combining everything together we have

$$\frac{d}{d\tau}\Big(\dot{x}^0 - 2h_{0\gamma}\dot{x}^\gamma + \zeta h_{00}\,\dot{q}\Big) = O(\varepsilon(x^0)^{-1})|\dot{X}|^2$$

We integrate this identity between proper times $0 < \tau$. Observe that $|\dot{X}| \leq |\dot{x}^0|$ and that

$$(x^0)^{-1}|\dot{x}^0| = \frac{d}{d\tau}\ln x^0$$

[16]This is the reason for introducing the cut-off function ζ.

Thus

$$\dot x^0(\tau) \lesssim \dot x^0(0) + (2h_{0\gamma}\dot x^\gamma - \zeta h_{00}\dot q)|_0^\tau + \varepsilon \int_0^\tau \frac{d}{d\tau} \ln x^0 \dot x^0 \, d\tau'$$

It follows that

$$\dot x^0(\tau) \lesssim \Big(\frac{|x^0(\tau)|}{|x^0(0)|}\Big)^\varepsilon \dot x^0(0)$$

Integrating one more time and assuming that $x^0(0) = t(0) = 1$ we obtain that

$$(x^0(\tau))^{1-\varepsilon} \lesssim 1 + \dot x^0(0)\tau$$

From this we conclude that the time $x^0 = t$ remains finite with τ. This concludes the proof for time-like geodesics. □

We now address the issue of null geodesics $X(\tau)$,

$$g^{\alpha\beta}\dot X^\alpha \dot X^\beta = 0$$

Examining the proof above leads to the conclusion that is suffices to establish that the condition $\dot x^0(\tau) > 0$ is preserved in time.

Lemma 16.3. *For a future oriented inextendible null geodesic $X(\tau)$ defined on the interval $[0,\tau_*)$ we have $\dot x^0(\tau) > 0$ for all $\tau \in [0,\tau_*)$.*

Proof. Let $\tau_0 < \tau_*$ be the first time when $\dot x^0(\tau_0) = 0$. Fix a sufficiently small small constant c. Then there exists a small interval of size δ such that

$$0 \le \dot x^0(\tau) \le c, \qquad \forall \tau \in [\tau_0 - \delta, \tau_0]$$

and

$$\dot x^0(\tau_0 - \delta) = c \tag{16.9}$$

We observe that (16.5) with $A = 0$ implies that $|\dot X(\tau)| \le 2|\dot x^0(\tau)|$ and therefore,

$$|\dot X(\tau)| \le 2c, \qquad \forall \tau \in [\tau_0 - \delta, \tau_0]$$

Integrating the geodesic equation (16.1) we obtain

$$|\dot x^0(\tau_0) - \dot x^0(\tau_0 - \delta)| \le \int_{\tau_0-\delta}^{\tau_0} |\Gamma||\dot X|^2 \le \varepsilon c^2 \delta$$

Thus, using (16.9),

$$\dot x^0(\tau_0) \ge c - \varepsilon c^2 \delta > 0$$

Contradiction. □

This completes the proof of Proposition 16.2.

We have shown that all future inextendible causal geodesics $X(\tau)$ exist for all values of the affine parameter $\tau \in [0,\infty)$. This means that the constructed space-time is future causally geodesically complete. Next we establish that all future oriented causal geodesics escape to infinity.

Proposition 16.4. *Let $X(\tau)$ be a future oriented causal geodesic. Then*

$$\lim_{\tau\to\infty}|X(\tau)| = \infty \tag{16.10}$$

Proof. The inequality (16.6) immediately gives the desired result for time-like geodesics. Recall that by Lemma 16.3 we have that $\dot{x}^0(\tau) > 0$ and thus $x^0(\tau)$ is monotonically increasing in τ. We now argue by contradiction. Assume that for all $\tau \geq 0$

$$|X(\tau)| \leq C$$

for some potentially large constant C. Then there exists a time t_0 such that

$$t_0 = \lim_{\tau\to\infty} x^0(\tau)$$

Set τ_0 be the value of the proper time τ for which $t(\tau_0) = t_0 - \delta$ for some small constant δ. Integrating the geodesic equation we obtain that for $\tau \geq \tau_0$

$$\dot{x}^0(\tau) = \dot{x}^0(\tau_0) + \int_{\tau_0}^{\tau} |\Gamma||\dot{x}^0|^2\, d\tau' \leq \dot{x}^0(\tau_0) + \varepsilon \int_{t_0-\delta}^{t} \dot{x}^0\, dt' \leq \dot{x}^0(\tau_0) + \varepsilon\delta \sup_{\tau_0\leq\tau'\leq\tau} \dot{x}^0(\tau) \tag{16.11}$$

Thus for any $\tau \geq \tau_0$

$$\dot{x}^0(\tau) \leq 2\dot{x}^0(\tau_0) \tag{16.12}$$

Choosing a sequence of times $\tau_0 \to \infty$ such that $\dot{x}^0(\tau_0) \to 0$ (such a sequence must exist, otherwise $x^0(\tau) \to \infty$) we infer from (16.12) that

$$\dot{x}^0(\tau) \to 0$$

as $\tau \to \infty$. We can then choose small constant c, δ such that $t(\tau_0) = t_0 - \delta$ and

$$\dot{x}^0(\tau_0) = c, \qquad \dot{x}^0(\tau) \leq c$$

for all $\tau \geq \tau_0$. Returning to (16.11) we see that

$$|\dot{x}^0(\tau) - c| \leq \varepsilon\delta c$$

Thus

$$\dot{x}^0(\tau) \geq \frac{c}{2}$$

for all $\tau \geq \tau_0$ and we obtained contradiction. □

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